Chapter 30

Section 1

We will now look at integration. Suppose we have the function $f(x) = x^2$. We call $f'(x)$ the derivative of $f(x)$.

The question we want to ask is "Can we undo this?". Let $f(x) = 2x$. Can we find a function $F(x)$ such that

$$F'(x) = f(x) = 2x?$$

Suppose $F(x) = x^2$, then

$$F'(x) = \frac{d}{dx}x^2 = 2x = f(x).$$

We call $F(x)$ an antiderivative of $f(x)$.

Why do we say **an** antiderivative and not **the** antiderivative? Lets look at the following table:

$F(x)$	$F'(x)$
x^2	$2x$
x^2+1	$2x$
x^2+2	$2x$
x^2+3	$2x$
$x^2+\pi$	$2x$
$x^2+10^{10^{10}}$	$2x$

So if $f(x) = 2x$, then

$$F(x) = x^2,\ F(x) = x^2+1,\ F(x) = x^2+\pi,\ \text{etc.}$$

are all antiderivatives of $f(x)$.

If C is any constant, then $F(x) = x^2 + C$ is an antiderivative of $f(x) = 2x$ since

$$F'(x) = \frac{d}{dx}(x^2+C) = 2x.$$

If $F(x)$ is an antiderivative of $f(x)$ we write

$$\int f(x)dx = F(x) + C$$

to indicate that we are finding an antiderivative. So

$$\int 2x dx = x^2 + C$$

We call

$$\int f(x)dx$$

the indefinite integral.

Note that if $f(x) = x$, then $f'(x) = 1$. So

$$\int 1 dx = \int dx = x + C$$

Next let us look at $f(x) = x^n$, so $f'(x) = nx^{n-1}$. For example, if $f(x) = x^2$, then $f'(x) = 2x^{2-1} = 2x$.

So to go from $f'(x) = 2x$ to $f(x) = x^2$ what should we do? Well if we divided by 2 and added 1 to the exponent we get

$$\frac{2x^{1+1}}{2} = x^2$$

Another example is $f(x) = x^3$ and $f'(x) = 3x^2$. This time we divide by 3 and again add 1 to the exponent:

$$\frac{3x^{2+1}}{3} = x^3$$

Note that in both examples we added one to the exponent and what we divided by was 1 added to the original exponent:

$$\frac{2x^{1+1}}{1+1} = \frac{2x^{1+1}}{2} = x^2, \qquad \frac{3x^{2+1}}{2+1} = \frac{3x^{2+1}}{3} = x^3$$

So in general if we have $f(x) = x^n$ and $f'(x) = nx^{n-1}$ to go from $f'(x)$ to $f(x)$ we add 1 to the exponent and divided by that number

$$\frac{nx^{(n-1)+1}}{(n-1)+1} = \frac{nx^n}{n} = x^n$$

So our rules thus far are

Theorem.

$$\int dx = \int 1 dx = x + C$$

Theorem.

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad \textit{where } n \neq -1$$

Example 1.

1. $\int x^2 dx =$

2. $\int x^4 dx =$

3. $\int dx =$

Another rule is

Theorem.

$$\int af(x)dx = a\int f(x)dx$$

Example 2.

1. $\int 2x dx =$

2. $\int 3 dx =$

3. $\int 10x^4 dx =$

Recall that $\frac{d}{dx}e^x = e^x$ and $\frac{d}{dx}\ln x = \frac{1}{x}$. This leads us to the following theorem:

Theorem.

$$\int e^x dx = e^x + C, \quad \int \frac{1}{x} dx = \int \frac{dx}{x} = \ln x + C$$

Example 3.

1. $\int 2e^x dx =$

2. $\int \frac{1}{x} dx =$

3. $\int \frac{3}{x} dx =$

Theorem.

$$\int f(x) + g(x)dx = \int f(x)dx + \int g(x)dx$$

Example 4.

$$\int x + 1dx =$$

Example 5.

$$\int 3x^2 + 2x^{-3} dx =$$

Example 6.

$$\int 2e^x - x^2 dx =$$

Example 7.

$$\int \frac{1}{x} - x^{-2} dx =$$

Example 8. *HW 2, 6, 24, 28, 36*

HW 30.1: 1-37 odd; EPS basic integrals

Section 2

Suppose we wanted to integrate $(x^2+1)^9(2x)$:

$$\int (x^2+1)^9(2x)dx$$

Clearly multiplying out $(x^2+1)^9(2x)$ is not feasible. So what should we do?

Let us try a trick. Let $u = x^2+1$. If we take the derivative of u then we have

$$\frac{du}{dx} = \frac{d}{dx}(x^2+1) = 2x$$

Multiplying both sides by dx we get

$$du = 2xdx$$

Now

$$\int (x^2+1)^9 2xdx = \int u^9 du = \frac{u^{10}}{10} + C = \frac{(x^2+1)^{10}}{10} + C$$

This is called integration by substitution.

Integration by substitution is the inverse of taking the derivative by the chain rule.

Let

$$f(x) = \frac{(x^2+1)^{10}}{10} + C$$

then

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\frac{(x^2+1)^{10}}{10} + C\right) \\ &= (x^2+1)^9 \frac{d}{dx}(x^2+1) \\ &= (x^2+1)^9(2x) \end{aligned}$$

That is exactly what we were trying to integrate above

$$\int (x^2+1)^9(2x)dx$$

Let us integrate

$$\int 2xe^{x^2}dx$$

We note that e^{x^2} is a composition of functions and its derivative is

$$\frac{d}{dx}e^{x^2} = e^{x^2}\frac{d}{dx}x^2 = e^{x^2}2x = 2xe^{x^2}$$

This tells us that integration by substitution should be tried. Now if $f(x) = e^{x^2}$ we can let $g(x) = e^x$ and $h(x) = x^2$. So $f(x) = g(h(x))$. The trick in substitution is to let u equal the inner function, $h(x)$. So we let

$$u = x^2$$

Then

$$\frac{du}{dx} = 2x \Rightarrow du = 2xdx$$

Hence

$$\int 2xe^{x^2}dx = \int e^{x^2}2xdx = \int e^u du = e^u + C = e^{x^2} + C$$

This leads us to the following theorem:

Theorem (Integration by Substitution). *If $g'(x)$ exists and $F(x)$ is an antiderivative of $f(x)$, then*

$$\int f(g(x))g'(x)dx = F(g(x)) + C.$$

If we let $u = g(x)$, then $du = g'(x)dx$ and

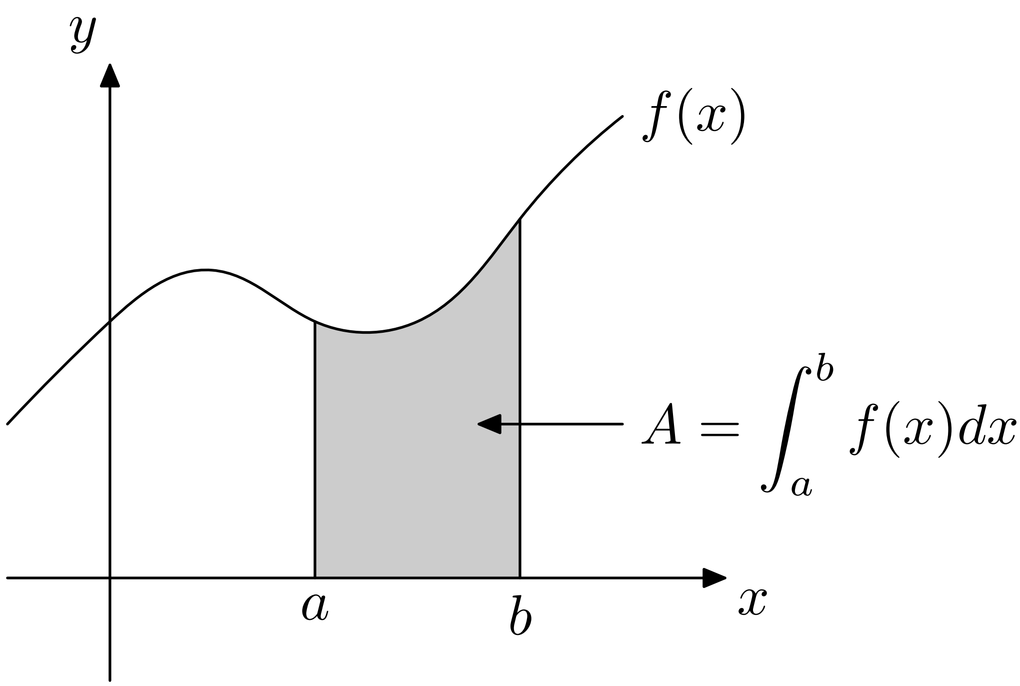
$$\int f(u)du = F(u) + C.$$

Example 9. *HW 2, 4, 8, 10, 12, 14, 16, 18, 20, 26, 28*

HW 30.2: 1-27 odd; EPS substitution

Section 4

How do we interpret an integral? We can interpret it geometrically as the area under the curve if we were told where to start and stop our measurement.



Note the new stuff in the integral symbol, the a and b. These are called the limits of integration and we call

$$\int_a^b f(x)dx$$

a definite integral.

How do we handle a definite integral?

Theorem (Fundamental Theorem of Calculus). *Let $f(x)$ be continuous and $F(x)$ its antiderivative, then*

$$\int_a^b f(x)dx = F(b) - F(a)$$

Before we demonstrate this theorem we need some new notation.

If U is some expression and a is a number, then the symbol $U\Big|_a$ tells us to plug the number a into the expression U.

Example 10.

1. $(x+1)\Big|_2 =$

2. $(x^2+x)\Big|_{-3} =$

Recall that

$$\int 2x dx = x^2 + C$$

So

$$\begin{aligned} \int_1^2 2x dx &= (x^2 + C)\Big|_2 - (x^2 + C)\Big|_1 \\ &= 2^2 + C - (1^2 + C) \\ &= 4 + C - 1 - C \\ &= 3 \end{aligned}$$

Note that the constant C canceled itself out. This will always happen. So we can avoid writing the constant:

$$\int_1^2 2x dx = x^2 \Big|_2 - x^2 \Big|_1 = 2^2 - 1^2 = 4 - 1 = 3$$

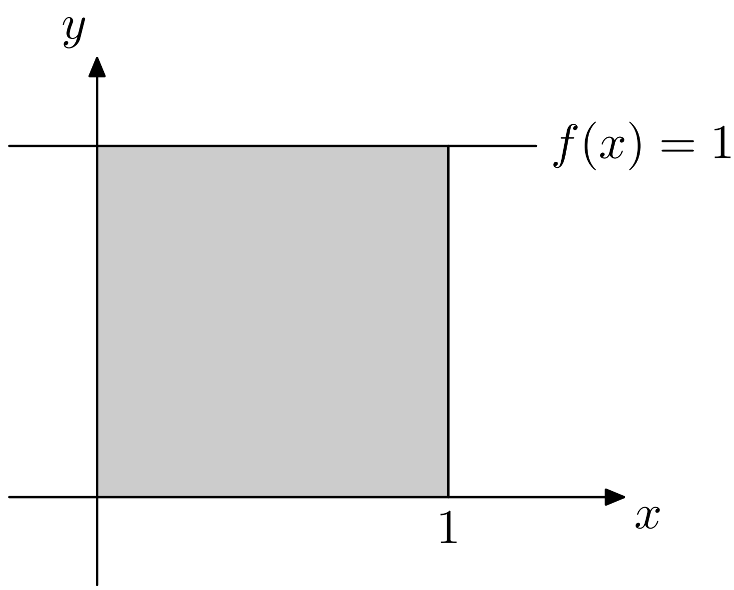
One more bit of notation is

$$x^2 \Big|_1^2 = x^2 \Big|_2 - x^2 \Big|_1$$

So

$$\int_1^2 2x dx = x^2 \Big|_1^2 = 2^2 - 1^2 = 3$$

Let us look at the idea that an integral is an area a little more. Suppose we have the function $f(x) = 1$ and we want to find the area under the curve from 0 to 1.



Clearly the area is

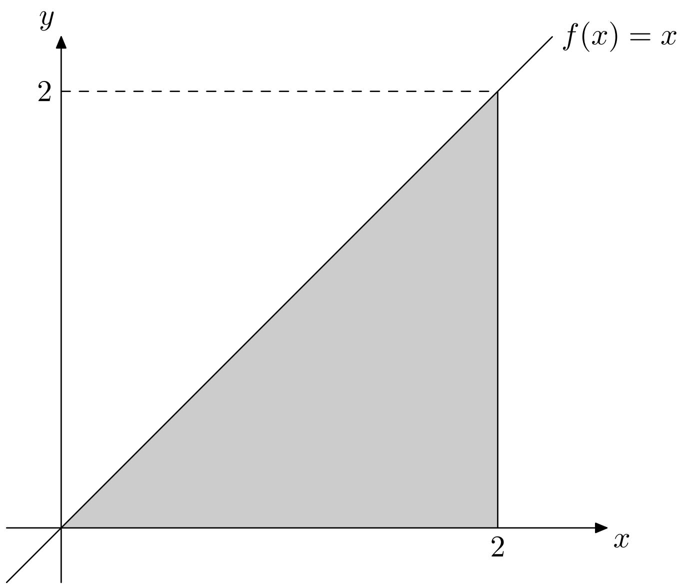
$$A = wh = (1)(1) = 1.$$

Now

$$\int_0^1 1 dx = x \Big|_0^1 = 1 - 0 = 1$$

So we see that the calculations agree.

Let us look at a slightly more complex calculation. Let $f(x) = x$ and find the area under the curve from 0 to 2.



This is a triangle, so the area is

$$A = \frac{1}{2}bh = \frac{1}{2}(2)(2) = 2$$

Using the integral gives us

$$\int_0^2 x\,dx = \frac{x^2}{2}\bigg|_0^2 = \frac{2^2}{2} - \frac{0^2}{2} = \frac{4}{2} = 2$$

Example 11. *HW: 2, 4, 8, 10, 12*

HW 30.4: 1-13 odd; EPS definite integral