

We are going to consider the derivative in one of its fundamental roles: the *slope of a tangent line*.

Since I can't do graphs easily here, I will refer to the figures on pages 228 and 229 of the book.

For the function f in the figure in the shaded box on p. 229, I want to figure out how to write the equation of the tangent line where $x = a$. In order to write the equation of a line, I need 2 facts: its slope and a point that is on the line. The point is known... It is $(a, f(a))$, so I need to find the slope of the imaginary tangent line that is shown as a blue line in the figure. So the problem becomes: How to find the slope of a line. Usually to find the slope of a line, I need 2 points; but how can we draw the tangent accurately enough to be sure that it is *the* tangent? (If I just guess at it, and pick 2 points, I will not be very accurate.... For example, if someone graphed a line such as $y = 3x - 1$, plotting the points $(-5, -16)$ and $(5, 14)$, could you look at the x -intercept and tell for sure that it was $\frac{1}{3}$? Probably not!)

So, here is the procedure: We move a "small" distance h away from a . This gives us an x -value of $a + h$. The point on f that corresponds to this is $(a + h, f(a + h))$. (Remember to find the y -value of an ordered pair, you substitute the given x -value. The y -value is then the same as " f of that x -value.") (See the figure). Now a line is drawn through our original point and the new point. This is the red line in the figure and is called a *secant* line because it intersects the graph in 2 points. Now I can easily find the slope of this secant line:

$$\text{Slope of secant line} = \frac{f(a + h) - f(a)}{(a + h) - a} \quad (\text{Difference of } y\text{-values divided by difference of } x\text{-values.})$$

$$\text{Notice that this reduces to: } \frac{f(a + h) - f(a)}{h}$$

Now we will use an intuitive idea of limits to see what happens:

Picture making h smaller.... This will move the point $(a + h, f(a + h))$ closer to our original point, but we will still have 2 points so that we can calculate slope. The idea of a few steps of this procedure are shown in the top figure on page 228. As h gets smaller and smaller, the secant line is a better and better approximation of the tangent line. The points are not labeled in this figure, because doing so would clutter up the picture so much that we wouldn't be able to let our intuition follow the progression of secant lines "approaching" the tangent line.

We find the slope of the tangent line at $x = a$ by:

$$\lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}$$

As we said in class, this value of h can be positive or negative, so it would make sense that we should get the same limit whether we approached 0 from the left or from the right in order to call this "the limit". You saw this demonstrated in problems 1 thru 4 of 3.5 where the sequences of numbers were approaching the same limit whether the values of h were positive or negative, as long as we let h get "smaller and smaller." the figure mentioned on page 229 only shows a picture of h approaching 0 from the right, that is h is positive and "small." To picture the limit as h approaches 0 from the left, that is h is negative and "small", we would have our point $(a + h, f(a + h))$ on the left side of $(a, f(a))$, and follow the same procedure. It is beyond the scope of this course to "prove" that these 2 limits exist and are equal for continuous functions. What is presented here is an intuitive idea of the process involved.

Using this limit, I can find the slope of the tangent line to most continuous functions at a given x -value a . However it is generally more convenient to find a “formula” that will calculate the slope of *any* tangent to f . To do this, we merely use an arbitrary x instead of a in the above formula and give it the name, $f'(x)$:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad \text{LIMIT DEFINITION OF DERIVATIVE}$$

Example 1: Given $f(x) = 5x + 3$, find $f'(x)$ using the above definition:

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[5(x+h) + 3] - [5x + 3]}{h} \quad (\text{Notice if I try “substitution”, I get } \frac{0}{0}, \text{ so I need to simplify.)} \\ &= \lim_{h \rightarrow 0} \frac{5x + 5h + 3 - 5x - 3}{h} \\ &= \lim_{h \rightarrow 0} \frac{5h}{h} \\ &= \lim_{h \rightarrow 0} 5 \end{aligned}$$

=5 (the limit of a constant function is that constant: Picture the “bug” crawling on a horizontal line... He is staying at that constant level and expects to stay there when he reaches his destination.)

Example 2: Given $f(x) = 3x^2 - 4$, find $f'(x)$ using the above definition:

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[3(x+h)^2 - 4] - [3x^2 - 4]}{h} \\ &= \lim_{h \rightarrow 0} \frac{[3(x^2 + 2xh + h^2) - 4] - [3x^2 - 4]}{h} \\ &= \lim_{h \rightarrow 0} \frac{3x^2 + 6xh + 3h^2 - 4 - 3x^2 + 4}{h} \\ &= \lim_{h \rightarrow 0} \frac{6xh + 3h^2}{h} \quad (\text{Notice at any step, if we try substitution, we get } \frac{0}{0}, \text{ so we factor and simplify:)} \\ &= \lim_{h \rightarrow 0} \frac{h(6x + 3h)}{h} \\ &= \lim_{h \rightarrow 0} (6x + 3h) \quad \text{Now we try substitution again and get:} \\ &= 6x. \end{aligned}$$

DON'T PANIC! We will NOT have to do every derivative by the definition this semester... There are shortcuts! However, learning this procedure should increase your understanding of the ideas of calculus.

SOME HINTS: Notice that in both of these examples that if we try substitution of $h = 0$ in the beginning, we get $\frac{0}{0}$: This will be true in general, so simplification is always needed. If the procedure is done correctly two things will always happen: In the numerator, all terms that do *not* have h as a factor will cancel out by addition. After this is done, h will be left as a common factor in the numerator, so you can factor and “cancel” out the h . This will allow you to use substitution to calculate the final step of the limit.

WHEN DERIVATIVES DO NOT EXIST: This can be considered from two viewpoints: One is very technical by considering the existence of “the limit” discussed above. As mentioned in class, the concept of limit can become difficult very quickly in some cases, so we will choose to look at the idea of the derivative not existing from an intuitive viewpoint.

Basically, I can use the illustration of the bug in his tiny corvette speeding along the curve that shows the graph of a given function. If there is a point on the graph where the bug’s tangent line trajectory would be one line if he were coming from the left, but would be a different trajectory if he were coming in from the right, THEN the derivative does NOT exist at that point. That is, the sequence of slopes of secant lines should be approaching the SAME number whether our h is positive or negative. A vivid illustration of this is $f(x) = |x|$ which is shown on page 252. The bug driving from the left is obviously on a straight line course with a slope of -1, while the bug driving from the right is on a straight line course with a slope of 1. The derivative DOES NOT EXIST at $x = 0$ for this function. (It does, however, exist everywhere else for this function.)

A rule of thumb to help us spot these “non-differentiable” points is to look for any place on the graph where there is a “sudden change of direction.” This would cause us to think that the tangent line would be one thing if we were moving in from the left, but we would think it would be different if we were moving in from the right. That is: There must be one and only one tangent line at a point for the derivative to exist there. Since the book apparently only mentions the example of the absolute value function, I will draw some graphs in class to illustrate other possibilities.

The other way that a derivative can fail to exist is for there to be one and only one tangent line but for its slope to be undefined. The only kind of line that has undefined slope is a vertical line, so we would be looking for functions that had points with vertical tangents. Two such functions are illustrated on page 261 in Figure 38. (Just think: If you rotated these graphs 90 degrees, they would appear to have horizontal tangent lines at the origin.)