2001 Comprehensive Test – Division II

1. What is the distance between the vertex of the parabola $y = x^2 - 4x + 3$ and the center of the circle $x^2 = 9 - (y-3)^2$?

(**A**) 4 (**B**) $2\sqrt{2}$ (**C**) $3\sqrt{2}$ (**D**) $2\sqrt{3}$ (**E**) $2\sqrt{5}$

2. For $x \geq 3$, $\sqrt{x^4 - 9x^2} = (?)$

(**A**) $x^2 - 3x$ (**B**) $x^2 + 3x$ (**C**) $x\sqrt{x^2-9}$ (**D**) $x^2\sqrt{x^2-9}$ (**E**) $3x\sqrt{x-1}$

3. Let $i = \sqrt{-1}$. Find the ninth term of the geometric sequence i, -2, $-4i$, ...

(**A**) -512 (**B**) $-256i$ (**C**) $16i$ (**D**) $128i$ (**E**) $256i$

4. Suppose s varies directly with t and inversely with r^2. If $s = 5$ when $r = 1$ and $t = 3$, what is the value of s when $r = \sqrt{3t}$?

(**A**) $\frac{1}{5}$ (**B**) $\frac{5}{9}$ (**C**) 5 (**D**) $5t^2$ (**E**) $\frac{\sqrt{3}}{t}$

5. $\dfrac{x^2 - 4(x+3)}{(x+2)(x+3)} = (?)$

(**A**) $x + 2$ (**B**) $x - 2$ (**C**) $\frac{x-2}{x+2}$ (**D**) $\frac{x-2}{x+3}$ (**E**) $\frac{x-6}{x+3}$

6. Find the sum of all solutions $x \in [-90°, 90°]$ for the equation $2\tan x \sin x + 2\sin x = \tan x + 1$.

(**A**) $-15°$ (**B**) $0°$ (**C**) $15°$ (**D**) $75°$ (**E**) $180°$

7. A natural number is *abundant* if it is less than the sum of its positive proper divisors (including 1, but not including the number itself). How many abundant numbers are less than 20?

(**A**) 1 (**B**) 2 (**C**) 3 (**D**) 4 (**E**) 5

8. $\sqrt{\sin^2 x + \csc^2 x + \cos^2 x + \sec^2 x - (\tan^2 x + \cot^2 x)} = (?)$

(**A**) $\frac{1}{2}$ (**B**) $\frac{\sqrt{3}}{2}$ (**C**) 1 (**D**) $\frac{2\sqrt{3}}{3}$ (**E**) $\sqrt{3}$

9. If the five digit number $5dddd$ is divisible by 6, what is the digit d?

(**A**) 2 (**B**) 4 (**C**) 6 (**D**) 7 (**E**) 8

10. $(x^{-1} + 3)^{-1} = (?)$

(**A**) $x + 3$ (**B**) $x + \frac{1}{3}$ (**C**) $\frac{x}{3}$ (**D**) $\frac{x}{1+3x}$ (**E**) $\frac{1+3x}{x}$

11. What is the cosine of the largest angle in a 3-4-6 triangle?

(**A**) $-\frac{3}{4}$ (**B**) $-\frac{11}{24}$ (**C**) 0 (**D**) $\frac{2}{3}$ (**E**) $\frac{29}{36}$

12. What is the diameter in feet of a pulley which is driven at 6 revolutions per second by a belt moving at 40 feet per second?

(**A**) $\frac{10}{3\pi}$ (**B**) $\frac{20}{3\pi}$ (**C**) $\frac{3\pi}{10}$ (**D**) $\frac{20\pi}{3}$ (**E**) $\frac{40\pi}{3}$

13. Let $i = \sqrt{-1}$. $(1-i)^8 = (?)$

(**A**) 8 (**B**) 16 (**C**) $8e^{i\pi/2}$ (**D**) $4\sqrt{2}e^{i\pi/2}$ (**E**) $16e^{i\pi/2}$

14. $\tan^{-1}\left(\frac{4}{3}\right) - \tan^{-1}\left(\frac{1}{7}\right) = (?)$

(**A**) $\frac{\pi}{6}$ (**B**) $\frac{\pi}{4}$ (**C**) $\frac{\pi}{2}$ (**D**) $\frac{\pi}{3}$ (**E**) 1

15. If $f(x) = x^2 - 2$, $f(x+2) = (?)$

(**A**) x^2 (**B**) $x^2 + 2$ (**C**) $x^2 + 4x + 2$

(**D**) $x^3 - 4$ (**E**) $x^3 + 2x^2 - 2x - 4$

16. Find the solution of $\sec x - 1 = \tan x$ with $x \in [0, 2\pi)$.

(**A**) 0 (**B**) $\frac{\pi}{2}$ (**C**) π (**D**) $\frac{4\pi}{3}$ (**E**) $\frac{3\pi}{2}$

17. What is the sum of the *squares* of the real and complex solutions of $x^4 + x^3 + 4x^2 + 4x = 0$?

(**A**) -7 (**B**) -3 (**C**) 0 (**D**) 1 (**E**) 9

18. How many integers satisfy the inequality $|x^2 - 8| < 4$?

(**A**) 0 (**B**) 1 (**C**) 2 (**D**) 3 (**E**) 4

19. If n is an integer greater than 3, then $\frac{n!(n-3)!}{(n-2)!(n-1)!} = (?)$

(**A**) $\frac{n}{n-2}$ (**B**) $\frac{n-3}{n-1}$ (**C**) $n!$

(**D**) $\frac{n}{(n-2)(n-1)!}$ (**E**) $\frac{(n^2-3n)!}{(n^2-3n+2)!}$

20. $\frac{\sin 2x}{\sin x} = (?)$

(**A**) 2 (**B**) $\sin 2$ (**C**) $\sec x$ (**D**) $\cos x$ (**E**) $2\cos x$

21. A circle has radius of length 5, is tangent to the line with equation $4x - 3y = 18$ at the point $(3, -2)$, and lies above the line. What is the equation of the circle?

(**A**) $x^2 - 14x + y^2 + 10y = -49$
(**B**) $x^2 - 6x + y^2 + 4y = 12$
(**C**) $x^2 + 2x + y^2 - 2y = 3$
(**D**) $x^2 + 2x + y^2 - 2y = 23$
(**E**) $x^2 + 6x + y^2 + 4y = 12$

22. $32^{-(\sqrt{5})^{-2}} = (?)$

(**A**) 32^6 (**B**) $\sqrt[3]{32}$ (**C**) 4 (**D**) $32^{2\sqrt{3}}$ (**E**) $\frac{1}{2}$

23. Find an equation of the perpendicular bisector of the segment connecting $A(-2, 3)$ and $B(6, -5)$.

(**A**) $x + y = -1$ (**B**) $x - y = -1$ (**C**) $x - y = 3$

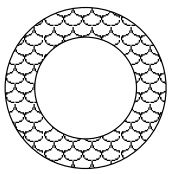
(**D**) $x + 2y = -4$ (**E**) $2x + y = 2$

24. Three circles, each with radius of length 6, intersect so that each circle passes through the centers of the other two circles. Find the area of the region that is the intersection of the interiors of the three circles.

(**A**) $9\sqrt{3}$ (**B**) 6π (**C**) 9π

(**D**) $18(\pi - \sqrt{3})$ (**E**) $48\pi - \sqrt{3}$

25. The diagram shows a circular pool surrounded by a garden of uniform width. The area of the garden is the same as the area of the surface of the pool. What is the ratio of the length of fencing needed to surround the pool to the length of fencing needed to enclose the entire region?

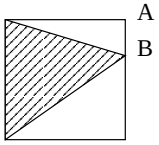


(**A**) $\frac{1}{4}$ (**B**) $\frac{\sqrt{2}}{4}$ (**C**) $\frac{1}{2}$ (**D**) $\frac{\sqrt{3}}{3}$ (**E**) $\frac{\sqrt{2}}{2}$

26. An investment with annual interest rate r yields an annual simple interest of \$1500. If \$500 more is invested and the rate is 2 percentage points less, the annual simple interest is \$1300. Find r.

(**A**) 7% (**B**) 8% (**C**) 9% (**D**) 11% (**E**) 12%

27. The diagram shows a square with sides of length 4. If $AB = x$, what is the ratio of the area of the shaded region to the total area of the unshaded regions?



(**A**) 1 (**B**) $\frac{1}{2}$ (**C**) $\frac{4}{x}$ (**D**) $2x$ (**E**) $\frac{4+x}{4-x}$

28. What is the probability that a 4-digit number consisting only of sixes and twos is divisible by 11?

(**A**) 0 (**B**) $\frac{1}{16}$ (**C**) $\frac{1}{8}$ (**D**) $\frac{3}{8}$ (**E**) $\frac{5}{16}$

29. Two distinct circles of radius 1 are tangent to each other. How many circles of radius 2 are tangent to both of them?

(**A**) 2 (**B**) 3 (**C**) 4 (**D**) 5 (**E**) 6

30. Find the final digit of $9^{412} \cdot 16^8$.

(**A**) 1 (**B**) 4 (**C**) 6 (**D**) 8 (**E**) 9

31. In a class of 100, 40 students study Spanish, 35 study German, and 27 study French. Two study all three subjects, while 3 study Spanish and German only, and 40 do not study either Spanish or French. Finally, 20 students study no language at all. How many students who study German do not study Spanish?

(**A**) 25 (**B**) 30 (**C**) 31 (**D**) 33 (**E**) 34

32. The lengths of the sides of a triangle are $x+5$, $2x-1$, and 4. For how many values of x will the triangle be isosceles?

(**A**) 1 (**B**) 2 (**C**) 3 (**D**) 4 (**E**) 5

33. Find the sum of the infinite series
$\frac{1}{7}+\frac{2}{7^2}+\frac{3}{7^3}+\frac{1}{7^4}+\frac{2}{7^5}+\frac{3}{7^6}+\ldots$
where the numerators repeat the cycle 1, 2, 3 every three terms.

(**A**) $\frac{49}{57}$ (**B**) $\frac{33}{172}$ (**C**) $\frac{49}{171}$ (**D**) $\frac{11}{57}$ (**E**) $\frac{11}{49}$

34. The length of rectangle $ABCD$ is twice its width w. P is a point such that the area of $\triangle PBD$ is equal to the area of the rectangle. What is the length of the altitude of $\triangle PBD$ to the base BD?

(**A**) $\frac{w}{2}$ (**B**) $\frac{2w\sqrt{5}}{5}$ (**C**) $\frac{2w\sqrt{3}}{3}$ (**D**) $\frac{4w\sqrt{5}}{5}$ (**E**) $\frac{4w\sqrt{3}}{3}$

35. What is the product of the solutions to the equation $ax^2+bx+c=0$, where $a\neq 0$?

(**A**) $4ac$ (**B**) $\frac{c}{2a}$ (**C**) $\frac{c}{a}$ (**D**) $\frac{b^2-2ac}{a}$ (**E**) $\frac{b^2+2ac}{a}$

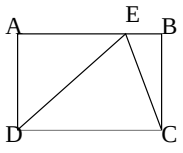
36. In $\triangle ABC$, $AB=AC$ and $m\angle ABC\leq 60°$. Which of the following is true?

(**A**) BC must be greater than or equal to the lengths of the other sides of $\triangle ABC$.
(**B**) BC must be less than or equal to the lengths of the other sides of $\triangle ABC$.
(**C**) $\triangle ABC$ must be equilateral.
(**D**) $\angle BAC$ must be acute.
(**E**) $\angle BAC$ must be obtuse.

37. How many solutions are there to the equation $t^2=|3t-2|$?

(**A**) 0 (**B**) 1 (**C**) 2 (**D**) 3 (**E**) 4

38. In the diagram, $ABCD$ is a rectangle, AB and BC are integers, $AE=4$, $EC=\sqrt{13}$, and the area of $\triangle DEC$ is 9. What is the perimeter of the rectangle?



(**A**) 6 (**B**) 9 (**C**) 14 (**D**) 18 (**E**) 22

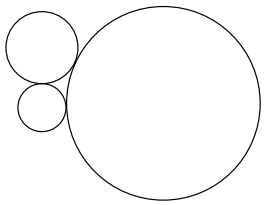
39. Rationalize the denominator and simplify: $\frac{2\sqrt{6}+\sqrt{3}}{\sqrt{6}-\sqrt{3}}=(?)$

(**A**) $1+\sqrt{2}$ (**B**) $3+6\sqrt{2}$ (**C**) $4+\sqrt{2}$ (**D**) $5+2\sqrt{2}$ (**E**) $5+3\sqrt{2}$

40. A person has a hours at his disposal. How many miles may he ride in a car traveling b miles per hour and yet have time to return on foot walking c miles per hour?

(**A**) $\frac{ab^2}{a+b}$ (**B**) $\frac{ab-ac}{ab+c}$ (**C**) $\frac{abc}{b+c}$ (**D**) $\frac{ac}{b+c}$ (**E**) $\frac{b+c}{ac}$

41. The diagram shows three mutually tangent circles. If the circles have radii 2, 3, and 7, respectively, what is the area of the triangle formed by the segments joining their centers.



(**A**) $\frac{45}{2}$ (**B**) $\sqrt{42}$ (**C**) $4\sqrt{3}$ (**D**) $6\sqrt{14}$ (**E**) 25

42. If the 3-digit number $4x3$ is added to 134, the result is the 3-digit number $5y7$, which is divisible by 7. Then $x + y = (?)$

(**A**) 5 (**B**) 7 (**C**) 9 (**D**) 11 (**E**) 13

43. Find a point on the y-axis that is equidistant from $(1, 1)$ and $(5, -5)$.

(**A**) $(0, -4)$ (**B**) $(0, -3)$ (**C**) $(0, -2)$ (**D**) $(0, 1)$ (**E**) $(3, -2)$

44. How many solutions of the equation $\cos x = \tan^{-1} 2x$ are in the interval $[-2\pi, 2\pi]$?

(**A**) 0 (**B**) 1 (**C**) 2 (**D**) 3 (**E**) 4

45. Exactly two integers between 75 and 85 are divisors of $3^{32} - 1$. What is the product of those integers?

(**A**) 5852 (**B**) 6560 (**C**) 6804 (**D**) 6888 (**E**) 6972

46. The lengths of the sides of $\triangle ABC$ are $AB = 7.5$, $BC = 10$, and $AC = 5$. Segment $\overline{BC}$ is extended through C to point P so that $\triangle PAB$ is similar to $\triangle PCA$. Then $CP = (?)$

(**A**) 7.5 (**B**) 8 (**C**) 10 (**D**) 12 (**E**) 15

47. What is the product of the solutions of $x^{\log_{10} x} = \frac{10000}{x^3}$?

(**A**) $\frac{1}{1000}$ (**B**) $\frac{1}{10}$ (**C**) 100 (**D**) 1000 (**E**) 1,000,000

48. If a committee of 6 members is to be chosen from among 5 Democrats and 3 Republicans so that at least two members of each party serve on the committee, how many committees are possible?

(**A**) 15 (**B**) 16 (**C**) 20 (**D**) 25 (**E**) 28

49. A container in the shape of a right circular cone of height 2 feet and radius r feet is full of water. A hole is punched in the vertex at the bottom of the cone, and the water drips into a second container in the shape of a right circular cylinder of the same radius. What is the depth (in feet) of the water in the cylinder when the volume of water in the cone is half of the original amount?

(**A**) $\frac{1}{3}$ (**B**) $\frac{2}{3}$ (**C**) 1 (**D**) $\frac{r}{3}$ (**E**) r

50. H, A, and L are positive integers such that $HAL = 2001$. If five *different* digits are required to write the numerals for H, A, and L, what is the largest of these integers?

(**A**) 29 (**B**) 69 (**C**) 79 (**D**) 87 (**E**) 667