2002 Comprehensive Test – Division I

1. If $a = 1$ and $b = -2$, then $a - b^{b-a} = (?)$

(**A**) -1 (**B**) $\frac{1}{27}$ (**C**) $\frac{7}{8}$ (**D**) 1 (**E**) $\frac{9}{8}$

2. If q varies directly as $\sqrt{h}$, and if h varies inversely as j^3, then q varies inversely as j^n. What is the value of n?

(**A**) -6 (**B**) $-\frac{3}{2}$ (**C**) $-\frac{2}{3}$ (**D**) $\frac{2}{3}$ (**E**) $\frac{3}{2}$

3. $\log(8.2 \times 10^{-9}) = (?)$

(**A**) -0.8 (**B**) $-9 + \log 8.2$ (**C**) $-9 \log 8.2$ (**D**) $\frac{\log 8.2}{10^9}$ (**E**) $\frac{\log 8.2}{9}$

4. Where defined, $\frac{A}{x+1} + \frac{B}{2x-1} = \frac{3(x-1)}{2x^2+x-1}$. Then $A + B = (?)$

(**A**) -3 (**B**) -2 (**C**) 0 (**D**) 1 (**E**) 3

5. If $x = 5 - \sqrt{y^2 - 25}$, then $x^2 = (?)$

(**A**) y^2 (**B**) $-y^2$ (**C**) $50 - y^2$ (**D**) $y^2 - 50$ (**E**) $y^2 - 10\sqrt{y^2 - 25}$

6. If $f(x) = a^{5x}$ and $a^3 = 16$, then $f(3) = (?)$

(**A**) 80 (**B**) 240 (**C**) $2^{20/9}$ (**D**) $2^{20/3}$ (**E**) 2^{20}

7. Find the coefficient of $x^{27}y^2$ in the expansion of $(x^3 + \sqrt{y})^{13}$.

(**A**) 29 (**B**) 78 (**C**) 286 (**D**) 715 (**E**) 1287

8. If $a \geq 0$, then $\sqrt{a\sqrt[3]{a\sqrt[4]{a}}} = (?)$

(**A**) $a\sqrt{a}$ (**B**) $\sqrt[8]{a}$ (**C**) $\sqrt[8]{a^3}$ (**D**) $\sqrt[24]{a}$ (**E**) $\sqrt[24]{a^{17}}$

9. How many solutions to the equation $\tan 3x = -\sqrt{3}$ are in the interval $(0, 2\pi)$?

(**A**) 0 (**B**) 1 (**C**) 2 (**D**) 4 (**E**) 6

10. Let the operation $*$ be defined by $a * b = a + b - ab$. Solve the equation $3 * (x * 2) = 14$.

(**A**) -12 (**B**) $-\frac{9}{5}$ (**C**) $\frac{7}{3}$ (**D**) $\frac{7}{2}$ (**E**) $\frac{15}{2}$

11. If the domain of $f(x)$ is $[0, 2]$, what is the domain of $2f(2x) - \frac{1}{2}f(\frac{1}{2}x + 2)$?

(**A**) $[0, 2]$ (**B**) $[0, 4]$ (**C**) $[\frac{1}{4}, 1]$ (**D**) $[2, 4]$ (**E**) $\{0\}$

12. If $\sqrt{x + 1.5} + \sqrt{x - 1.5} = 3$, then $\sqrt{x + 1.5} - \sqrt{x - 1.5} = (?)$

(**A**) $-\sqrt{3}$ (**B**) 0 (**C**) 1 (**D**) $\sqrt{3}$ (**E**) 2

13. Let N be a perfect square. Let x and y be positive integers such that $xy = N$ and the greatest common divisor of x and y is 1. Which of the following must always be true?

(**A**) Both x and y are perfect squares.
(**B**) Exactly one of x and y is a perfect square.
(**C**) The sum $x + y$ is a perfect square.
(**D**) The positive difference $|x - y|$ is a perfect square.
(**E**) None of these

14. The function $f(x) = -x^2 - 6x - 3$ has

(**A**) minimum value -3.
(**B**) minimum value 6.
(**C**) minimum value 24.
(**D**) maximum value 6.
(**E**) maximum value 24.

15. $\sin^3\theta + \cos^3\theta = (?)$

(**A**) $\cos 3\theta$
(**B**) $3\sin^2\theta\cos\theta + 3\sin\theta\cos^2\theta$
(**C**) $(\sin\theta + \cos\theta)(1 - \sin\theta\cos\theta)$
(**D**) $(\sin\theta + \cos\theta)(\sin^2\theta - \cos^2\theta)$
(**E**) $(\sin\theta + \cos\theta)^3 - 3\sin\theta\cos\theta$

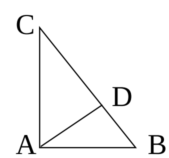
16. If $f(x) = \dfrac{x+1}{x-1}$, then $f^{-1}(x^{-1}) = (?)$

(**A**) 1 (**B**) $\dfrac{x+1}{x-1}$ (**C**) $\dfrac{x-1}{x+1}$ (**D**) $\dfrac{x+1}{1-x}$ (**E**) $\dfrac{1-x}{x+1}$

17. How many solutions to the equation $\cos^2\theta = \cos\theta$ are in the interval $0 \le \theta \le 2\pi$?

(**A**) 0 (**B**) 1 (**C**) 2 (**D**) 3 (**E**) 4

18. In the diagram, $AB = 3$, $AC = 4$, $BC = 5$, and $\overline{AD}$ is the altitude to the hypotenuse $\overline{BC}$. Find the ratio of the perimeter of $\triangle ABD$ to the perimeter of $\triangle ADC$.



(**A**) $\dfrac{9}{16}$ (**B**) $\dfrac{3}{5}$ (**C**) $\dfrac{3}{4}$ (**D**) $\dfrac{9}{5}$ (**E**) $\dfrac{16}{5}$

19. Find the length of a diagonal of a regular pentagon whose sides have length 1.

(**A**) $2\cos(\pi/5)$ (**B**) $\sqrt{2}$ (**C**) $\sqrt{2} + \cos(\pi/5)$
(**D**) $\sqrt{2}(1 + \cos(\pi/5))$ (**E**) $\sqrt{2}(1 + \cos(2\pi/5))$

20. If $x \neq y$ and $x, y > 0$, then $\dfrac{\sqrt[3]{64(x^{18}y^6)^{1/2}}}{\left(\dfrac{xy^{-1} - 1}{x^{-2}y^{-1} - x^{-3}}\right)} = (?)$

(**A**) $4y$ (**B**) $2y\sqrt{2}$ (**C**) $\dfrac{x}{y^3}$ (**D**) $\dfrac{4x}{y^2}$ (**E**) $\dfrac{xy^2}{4}$

21. When the radius r of the base of a right circular cone is increased by n units, the volume of the cone is doubled. What is the value of n?

(**A**) r (**B**) $2r$ (**C**) $r(\sqrt{2} - 1)$ (**D**) $r(\sqrt{2} + 1)$ (**E**) $r(2 - \sqrt{2})$

22. $\dfrac{1}{\cot\alpha - \cot\beta} = (?)$

(A) $\tan\alpha - \tan\beta$ **(B)** $\dfrac{\sin\alpha\sin\beta}{\sin(\beta-\alpha)}$ **(C)** $\cos\alpha\cos\beta$

(D) $\dfrac{\cos\alpha\cos\beta}{\cos^2(\alpha-\beta)}$ **(E)** $\dfrac{\cos\alpha\cos\beta}{\sin(\alpha+\beta)}$

23. Which of the following is one of the factors of $x^4 + 4$?

(A) $x+2$ **(B)** x^2+2 **(C)** x^2-2 **(D)** x^2-2x+2 **(E)** x^2-4x+2

24. Find the remainder when 4^{19} is divided by 7.

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

25. Find the number of distinguishable permutations of 10 coins if 4 are pennies, 3 are nickels, 2 are dimes, and one is a quarter.

(A) 1 **(B)** 720 **(C)** 5040 **(D)** 12,600 **(E)** 1,134,000

26. Find the sum of the solutions of the equation $\begin{vmatrix} x & 3 & 1 \\ 2 & 5 & 0 \\ 3 & x^2 & -2 \end{vmatrix} = 9.$

(A) 0 **(B)** 1 **(C)** 5 **(D)** 8 **(E)** 14

27. What is the maximum value of the function $f(x) = \dfrac{x^2}{x^2-x-2}$ on the interval $(-1, 2)$?

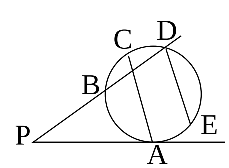
(A) $-\dfrac{9}{4}$ **(B)** $-\dfrac{1}{2}$ **(C)** 0

(D) $\dfrac{1}{2}$ **(E)** there is none

28. Let $i = \sqrt{-1}$. Then $(\sqrt{3} - i)^{36} = (?)$

(A) -2^{36} **(B)** 2^{36} **(C)** -2^{18} **(D)** $2^{18}(\sqrt{3}-1)$ **(E)** $2^{16}(\sqrt{3}+1)$

29. In the diagram, $\overleftrightarrow{PA}$ is tangent to the circle at A and $\overleftrightarrow{PD}$ intersects the circle at B and D. If $\overleftrightarrow{CA} \parallel \overleftrightarrow{DE}$, $\widehat{BC} \cong \widehat{CD}$, $m\,\widehat{DE} = 140°$, and $m\,\widehat{AE} = 30°$, then $m\angle APB = (?)$



(A) 15° **(B)** 20° **(C)** 30° **(D)** 40° **(E)** 60°

30. Determine the largest possible integer n such that 942! is divisible by 15^n.

(A) 62 **(B)** 125 **(C)** 233 **(D)** 314 **(E)** 471

31. Suppose a and b are constants such that the system $x^2 - y = a$ and $y - x = b$ has exactly one solution. What is the x-value in the solution?

(A) $-\dfrac{1}{2}$ **(B)** 0 **(C)** $\dfrac{1}{2}$ **(D)** 1 **(E)** $\dfrac{3}{2}$

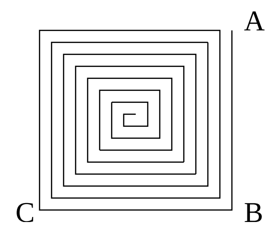
32. A lattice point is a point in the plane with integer coordinates. How many lattice points are on the line segment whose endpoints are $(\frac{1}{2}, \frac{1}{3})$ and $(100\frac{1}{2}, 100\frac{1}{3})$?

(A) 0 **(B)** 4 **(C)** 16 **(D)** 50 **(E)** 100

33. Find the solution set for the inequality $\dfrac{(2x+1)^2(x-1)}{x(x^2-1)} \geq 0$.

(**A**) $(-\infty,-1)\cup(-1,\infty)$ (**B**) $(-\infty,-1)\cup(0,\infty)$ (**C**) $(-\infty,-1)\cup(0,1)\cup(1,\infty)$ (**D**) $(-\frac{1}{2},0)\cup(0,1)\cup(1,\infty)$ (**E**) $(-\infty,-1)\cup\{-\frac{1}{2}\}\cup(0,1)\cup(1,\infty)$

34. The diagram shows a maze in which the horizontal and vertical segments are spaced one unit apart. Thus $AB = 15$, $BC = 16$, and the horizontal segment at the center of the maze has length 1. What is the total length of the spiral path forming the maze?



(**A**) 256 (**B**) 270 (**C**) 271 (**D**) 272 (**E**) 288

35. There are 25 Walkmans at Electronics-R-Us. The manager knows that four Walkmans have defects, but it is not known which ones have the problem. Which of the following is closest to the chance of choosing three defect-free Walkmans out of the 25?

(**A**) 28% (**B**) 44% (**C**) 58% (**D**) 63% (**E**) 72%

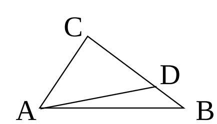
36. A circle is inscribed in a regular twelve-sided polygon. What is the ratio of the area of the polygon to the area of the circle?

(**A**) $\dfrac{6\tan 15°}{\pi}$ (**B**) $\dfrac{12\tan 15°}{\pi}$ (**C**) $\dfrac{18\tan 15°}{\pi}$ (**D**) $\dfrac{24\tan 15°}{\pi}$ (**E**) $\dfrac{36\tan 15°}{\pi}$

37. Express the polar equation $r^2 = -4r\cos\theta$ as a cartesian equation.

(**A**) $x^2+y^2=4$ (**B**) $(x+2)^2+y^2=4$ (**C**) $(x-2)^2+y^2=4$ (**D**) $x^2+(y+2)^2=4$ (**E**) $x^2+(y-2)^2=4$

38. In the diagram, $AC = CD$ and $m\angle CAB - m\angle ABC = 30°$. What is $m\angle BAD$?



(**A**) 10° (**B**) 15° (**C**) 20° (**D**) $22\frac{1}{2}°$ (**E**) 30°

39. The base ten fraction $\dfrac{227}{342}$ can be represented as $0.443\overline{443}$ in base B. What is the value of B?

(**A**) 5 (**B**) 6 (**C**) 7 (**D**) 8 (**E**) 9

40. Let n be a positive even integer. What is the largest possible number of primes in the set $\{n!+1, n!+2, n!+3, \ldots, n!+n\}$?

(**A**) 0 (**B**) 1 (**C**) 2 (**D**) $\dfrac{n-2}{2}$ (**E**) $\dfrac{n}{2}$

41. A sequence of sets, $A_1 = \{1\}$, $A_2 = \{\frac{1}{2}, \frac{1}{4}\}$, $A_3 = \{\frac{1}{8}, \frac{1}{16}, \frac{1}{32}\}$, …, is formed from consecutive terms of the geometric sequence with common ratio $\frac{1}{2}$ so that each set A_n has one more element than the preceeding set A_{n-1}. What is the sum of the elements in the set A_8?

(**A**) $\dfrac{2^8-1}{2^{35}}$ (**B**) $\dfrac{2^9-1}{2^{35}}$ (**C**) $\dfrac{2^8-1}{2^{34}}$ (**D**) $\dfrac{2^9-1}{2^{34}}$ (**E**) $\dfrac{2^7+1}{2^{33}}$

42. $\cos^{-1}\left(\frac{2\sqrt{5}}{5}\right) + \cos^{-1}\left(\frac{3\sqrt{10}}{10}\right) = (?)$

(**A**) $\frac{\pi}{4}$ (**B**) $\frac{\pi}{3}$ (**C**) $\frac{\pi}{2}$ (**D**) $\frac{3\pi}{4}$ (**E**) $\frac{5\pi}{6}$

43. For each point (x, y) on an ellipse, the sum of the distances from (x, y) to the points $(\pm 2, 0)$ is 8. Find the positive value of x such that $(x, 3)$ is on the ellipse.

(**A**) 0 (**B**) $\frac{\sqrt{3}}{3}$ (**C**) 2 (**D**) 4 (**E**) $2\sqrt{3}$

44. If $x^2 + y^2 = 14xy$ and $\log[k(x+y)] = \frac{1}{2}(\log x + \log y)$, then $k = (?)$

(**A**) $\frac{1}{16}$ (**B**) $\frac{1}{14}$ (**C**) $\frac{1}{4}$ (**D**) $\frac{\log 14}{2}$ (**E**) $2\log 2$

45. When $x^{105} - 2$ is divided by $x - 1$, what is the remainder?

(**A**) -3 (**B**) -1 (**C**) 0 (**D**) 1 (**E**) 2

46. $(\log_{3y} 9^{y/2})(1 + \log_3 y) = (?)$

(**A**) y (**B**) $y + 3$ (**C**) $y + 9y\log_3 y$

(**D**) $1 + y\log_{3y} 3$ (**E**) $y\log_{3y}(3y+1)$

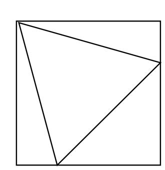
47. The equation $3x^4 - 7x^3 + 5x^2 - 7x + 2 = 0$ has exactly two rational roots, both of which are positive. Find the sum of those roots.

(**A**) $\frac{4}{3}$ (**B**) $\frac{5}{3}$ (**C**) $\frac{7}{3}$ (**D**) $\frac{8}{3}$ (**E**) 3

48. Let S_N be the sum $r_1 + r_2 + r_3 + \ldots$ of the exponents in the prime factorization $N = 2^{r_1} \cdot 3^{r_2} \cdot 5^{r_3} \cdot \ldots$ of a natural number N. Find all possible values of S_N where N is a perfect square when divided by 2 and a perfect cube when divided by 3.

(**A**) $S_N = 2k$, for $k = 3, 4, 5, \ldots$
(**B**) $S_N = 6k$, for $k = 1, 2, 3, \ldots$
(**C**) $S_N = 6k + 1$, for $k = 1, 2, 3, \ldots$
(**D**) $S_N = 6k - 1$, for $k = 1, 2, 3, \ldots$
(**E**) $S_N = 8k - 1$, for $k = 1, 2, 3, \ldots$

49. The diagram shows an equilateral triangle inscribed in a square. What is the ratio of the area of the triangle to the area of the square?



(**A**) $\frac{1}{2}$ (**B**) $\frac{\sqrt{3}}{4}$ (**C**) $2\sqrt{3} - 3$

(**D**) $\frac{3-\sqrt{3}}{2}$ (**E**) $\frac{\sqrt{4+2\sqrt{3}}}{4}$

50. 2002 is a palindrome, meaning that the number is unchanged when the order of the digits is reversed. How many proper factors of 2002 are palindromes consisting of more than one digit?

(**A**) 1 (**B**) 2 (**C**) 3 (**D**) 4 (**E**) 5