2002 Comprehensive Test – Division III

1. If $a = 1$ and $b = -2$, then $a - b^{b-a} = (?)$

(**A**) -1 (**B**) $\frac{1}{27}$ (**C**) $\frac{7}{8}$ (**D**) 1 (**E**) $\frac{9}{8}$

2. If q varies directly as $\sqrt{h}$, and if h varies inversely as j^3, then q varies inversely as j^n. What is the value of n?

(**A**) -6 (**B**) $-\frac{3}{2}$ (**C**) $-\frac{2}{3}$ (**D**) $\frac{2}{3}$ (**E**) $\frac{3}{2}$

3. $\log(8.2 \times 10^{-9}) = (?)$

(**A**) -0.8 (**B**) $-9 + \log 8.2$ (**C**) $-9 \log 8.2$ (**D**) $\frac{\log 8.2}{10^9}$ (**E**) $\frac{\log 8.2}{9}$

4. $\frac{2}{x+1} + \frac{3}{x-1} = (?)$

(**A**) 5 (**B**) $\frac{5}{2x}$ (**C**) $\frac{5}{x^2-1}$ (**D**) $\frac{5x}{x^2-1}$ (**E**) $\frac{5x+1}{x^2-1}$

5. Where defined, $\frac{A}{x+1} + \frac{B}{2x-1} = \frac{3(x-1)}{2x^2+x-1}$. Then $A + B = (?)$

(**A**) -3 (**B**) -2 (**C**) 0 (**D**) 1 (**E**) 3

6. If $x = 5 - \sqrt{y^2 - 25}$, then $x^2 = (?)$

(**A**) y^2 (**B**) $-y^2$ (**C**) $50 - y^2$ (**D**) $y^2 - 50$ (**E**) $y^2 - 10\sqrt{y^2 - 25}$

7. If $f(x) = a^{5x}$ and $a^3 = 16$, then $f(3) = (?)$

(**A**) 80 (**B**) 240 (**C**) $2^{20/9}$ (**D**) $2^{20/3}$ (**E**) 2^{20}

8. Find an equation for the perpendicular bisector of the line segment connecting the points $(-2, 5)$ and $(3, -7)$.

(**A**) $10x - 24y = 29$ (**B**) $10x + 24y = -19$ (**C**) $12x + 5y = 1$ (**D**) $24x - 10y = 15$ (**E**) $24x + 10y = -5$

9. If $a \geq 0$, then $\sqrt{a \sqrt[3]{a \sqrt[4]{a}}} = (?)$

(**A**) $a\sqrt{a}$ (**B**) $\sqrt[8]{a}$ (**C**) $\sqrt[8]{a^3}$ (**D**) $\sqrt[24]{a}$ (**E**) $\sqrt[24]{a^{17}}$

10. How many solutions to the equation $\tan 3x = -\sqrt{3}$ are in the interval $(0, 2\pi)$?

(**A**) 0 (**B**) 1 (**C**) 2 (**D**) 4 (**E**) 6

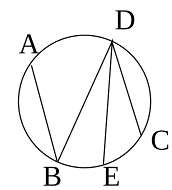
11. $4^{-3/2} = (?)$

(**A**) -8 (**B**) -6 (**C**) $\frac{1}{8}$ (**D**) $\frac{1}{6}$ (**E**) 6

12. Let the operation $*$ be defined by $a * b = a + b - ab$. Solve the equation $3 * (x * 2) = 14$.

(**A**) -12 (**B**) $-\frac{9}{5}$ (**C**) $\frac{7}{3}$ (**D**) $\frac{7}{2}$ (**E**) $\frac{15}{2}$

13. The diagram shows a circle. $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, $m\angle ABD = 60°$, and $\overrightarrow{DE}$ bisects $\angle BDC$. Then $m\,\widehat{CE} = (?)$



(**A**) $15°$ (**B**) $30°$ (**C**) $60°$ (**D**) $90°$ (**E**) $120°$

14. How many real roots does the polynomial $p(x) = 2x^3 + 5x^2 + 9x$ have?

(**A**) 0 (**B**) 1 (**C**) 2 (**D**) 3 (**E**) 4

15. Find the y-intercepts of the circle having center $C(4, 5)$ and radius 5.

(**A**) $(0, -5)$ and $(0, 4)$ (**B**) $(0, -2)$ and $(0, 3)$ (**C**) $(0, 0)$ and $(0, 5)$

(**D**) $(0, 1)$ and $(0, 6)$ (**E**) $(0, 2)$ and $(0, 8)$

16. The function $f(x) = -x^2 - 6x - 3$ has

(**A**) minimum value -3.
(**B**) minimum value 6.
(**C**) minimum value 24.
(**D**) maximum value 6.
(**E**) maximum value 24.

17. $\sin^3\theta + \cos^3\theta = (?)$

(**A**) $\cos 3\theta$
(**B**) $3\sin^2\theta\cos\theta + 3\sin\theta\cos^2\theta$
(**C**) $(\sin\theta + \cos\theta)(1 - \sin\theta\cos\theta)$
(**D**) $(\sin\theta + \cos\theta)(\sin^2\theta - \cos^2\theta)$
(**E**) $(\sin\theta + \cos\theta)^3 - 3\sin\theta\cos\theta$

18. If $f(x) = x^2 - 1$, then $f(x + 1) = (?)$

(**A**) x^2 (**B**) $x^2 + 2x$ (**C**) $x^2 - 2x$

(**D**) $x^2 + x$ (**E**) $x^3 + x^2 - x - 1$

19. Find the domain of the function $f(x) = \sqrt{18 - 2x^2}$.

(**A**) $[-3, 3]$ (**B**) $[0, 3]$ (**C**) $(-\infty, 3]$ (**D**) $[-3, \infty)$ (**E**) $(-\infty, \infty)$

20. Find the smallest whole number B so that 61 can be expressed in base B using only two digits.

(**A**) 6 (**B**) 7 (**C**) 8 (**D**) 9 (**E**) 10

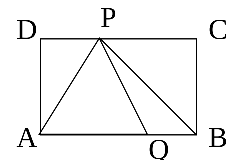
21. Find the solution set for the inequality $|6x - 5| \geq 10$.

(**A**) $(-\infty, -\frac{5}{6}]$ (**B**) $(-\infty, -\frac{5}{2}] \cup [\frac{5}{6}, \infty)$ (**C**) $[\frac{5}{2}, \infty)$

(**D**) $[-\frac{5}{6}, \frac{5}{2}]$ (**E**) $(-\infty, -\frac{5}{6}] \cup [\frac{5}{2}, \infty)$

22. Find the length of a diagonal of a regular pentagon whose sides have length 1.

(**A**) $2\cos(\pi/5)$ (**B**) $\sqrt{2}$ (**C**) $\sqrt{2}+\cos(\pi/5)$ (**D**) $\sqrt{2}(1+\cos(\pi/5))$ (**E**) $\sqrt{2}(1+\cos(2\pi/5))$



23. In the diagram, $ABCD$ is a rectangle and $\triangle APQ$ is equilateral. If $m\angle QPB = 30°$, then $m\angle PBC = (?)$

(**A**) $30°$ (**B**) $45°$ (**C**) $60°$ (**D**) $70°$ (**E**) $75°$

24. When the radius r of the base of a right circular cone is increased by n units, the volume of the cone is doubled. What is the value of n?

(**A**) r (**B**) $2r$ (**C**) $r(\sqrt{2}-1)$ (**D**) $r(\sqrt{2}+1)$ (**E**) $r(2-\sqrt{2})$

25. Let $i=\sqrt{-1}$. Compute $(3-2i)^3$.

(**A**) $-35+i$ (**B**) $-9+46i$ (**C**) $-i$ (**D**) $27-8i$ (**E**) $39-26i$

26. A store is having a 30% off sale. A CD player is on sale for \$203. What would the same CD player sell for if it were on sale at only 20% off?

(**A**) \$210.12 (**B**) \$213 (**C**) \$223.30 (**D**) \$232 (**E**) \$243

27. $\dfrac{2(x-3)+4}{(x-3)(x-1)} = (?)$

(**A**) $\dfrac{1}{x-3}$ (**B**) $\dfrac{2}{x-3}$ (**C**) $\dfrac{4}{x-1}$ (**D**) $\dfrac{6}{x-1}$ (**E**) $\dfrac{2x+1}{2x-4}$

28. Find the number of distinguishable permutations of 10 coins if 4 are pennies, 3 are nickels, 2 are dimes, and one is a quarter.

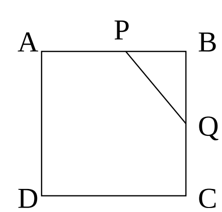
(**A**) 1 (**B**) 720 (**C**) 5040 (**D**) 12,600 (**E**) 1,134,000

29. Find the sum of the solutions of the equation $\begin{vmatrix} x & 3 & 1 \\ 2 & 5 & 0 \\ 3 & x^2 & -2 \end{vmatrix} = 9.$

(**A**) 0 (**B**) 1 (**C**) 5 (**D**) 8 (**E**) 14

30. Find the area of a rhombus whose diagonals have lengths 6 and 9.

(**A**) 13.5 (**B**) 27 (**C**) 40.5 (**D**) 48 (**E**) 54



31. In the diagram, $ABCD$ is a square with side of length 8. If $AP=5$ and $CQ=4$, what is the area of $APQCD$?

(**A**) 40 (**B**) 44 (**C**) 52 (**D**) 54 (**E**) 58

32. Determine the largest possible integer n such that 942! is divisible by 15^n.

(**A**) 62 (**B**) 125 (**C**) 233 (**D**) 314 (**E**) 471

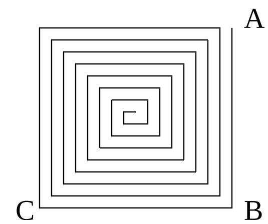
33. Suppose a and b are constants such that the system $x^2 - y = a$ and $y - x = b$ has exactly one solution. What is the x-value in the solution?

(**A**) $-\frac{1}{2}$ (**B**) 0 (**C**) $\frac{1}{2}$ (**D**) 1 (**E**) $\frac{3}{2}$

34. A lattice point is a point in the plane with integer coordinates. How many lattice points are on the line segment whose endpoints are $(\frac{1}{2}, \frac{1}{3})$ and $(100\frac{1}{2}, 100\frac{1}{3})$?

(**A**) 0 (**B**) 4 (**C**) 16 (**D**) 50 (**E**) 100

35. The diagram shows a maze in which the horizontal and vertical segments are spaced one unit apart. Thus $AB = 15$, $BC = 16$, and the horizontal segment at the center of the maze has length 1. What is the total length of the spiral path forming the maze?



(**A**) 256 (**B**) 270 (**C**) 271 (**D**) 272 (**E**) 288

36. There are 25 Walkmans at Electronics-R-Us. The manager knows that four Walkmans have defects, but it is not known which ones have the problem. Which of the following is closest to the chance of choosing three defect-free Walkmans out of the 25?

(**A**) 28% (**B**) 44% (**C**) 58% (**D**) 63% (**E**) 72%

37. $\log_b 8 + \log_b 2 = (?)$

(**A**) $\log_b 2$ (**B**) $2\log_b 2$ (**C**) $4\log_b 2$ (**D**) $5\log_b 2$ (**E**) $\log_b 10$

38. $(1 + x^{-2})^{-1} = (?)$

(**A**) $1 + x^2$ (**B**) $1 - x^2$ (**C**) $x^2 - 1$ (**D**) $\frac{x^2}{1 + x^2}$ (**E**) $\frac{1 + x^2}{x^2}$

39. Find the coefficient of $x^9 y$ in the expansion of $(2y - 4x^3)^4$.

(**A**) -512 (**B**) -64 (**C**) -32 (**D**) 128 (**E**) 256

40. Let n be a positive even integer. What is the largest possible number of primes in the set $\{n! + 1, n! + 2, n! + 3, \ldots, n! + n\}$?

(**A**) 0 (**B**) 1 (**C**) 2 (**D**) $\frac{n-2}{2}$ (**E**) $\frac{n}{2}$

41. Suppose $f(x) = \sqrt[3]{\frac{2x - 1}{x}}$. Calculate $f^{-1}(-1)$.

(**A**) -1 (**B**) 0 (**C**) $\frac{1}{3}$ (**D**) $\sqrt[3]{2}$ (**E**) $\sqrt[3]{3}$

42. $\cos^{-1}\left(\frac{2\sqrt{5}}{5}\right) + \cos^{-1}\left(\frac{3\sqrt{10}}{10}\right) = (?)$

(**A**) $\frac{\pi}{4}$ (**B**) $\frac{\pi}{3}$ (**C**) $\frac{\pi}{2}$ (**D**) $\frac{3\pi}{4}$ (**E**) $\frac{5\pi}{6}$

43. For each point (x, y) on an ellipse, the sum of the distances from (x, y) to the points $(\pm 2, 0)$ is 8. Find the positive value of x such that $(x, 3)$ is on the ellipse.

(**A**) 0 (**B**) $\frac{\sqrt{3}}{3}$ (**C**) 2 (**D**) 4 (**E**) $2\sqrt{3}$

44. If a function f satisfies the equation $f(a+b) = f(a) + f(b)$ for every pair of real numbers a and b, then what is (are) the possible value(s) of $f(0)$?

(**A**) 0 and 1 only (**B**) 0 only (**C**) 1 only (**D**) any real number (**E**) any positive real number

45. When $x^{105} - 2$ is divided by $x - 1$, what is the remainder?

(**A**) -3 (**B**) -1 (**C**) 0 (**D**) 1 (**E**) 2

46. $-2^3 \cdot 2^{-3} = (?)$

(**A**) -64 (**B**) -1 (**C**) 0 (**D**) 1 (**E**) 64

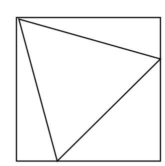
47. The equation $3x^4 - 7x^3 + 5x^2 - 7x + 2 = 0$ has exactly two rational roots, both of which are positive. Find the sum of those roots.

(**A**) $\frac{4}{3}$ (**B**) $\frac{5}{3}$ (**C**) $\frac{7}{3}$ (**D**) $\frac{8}{3}$ (**E**) 3

48. Let S_N be the sum $r_1 + r_2 + r_3 + \ldots$ of the exponents in the prime factorization $N = 2^{r_1} \cdot 3^{r_2} \cdot 5^{r_3} \cdot \ldots$ of a natural number N. Find all possible values of S_N where N is a perfect square when divided by 2 and a perfect cube when divided by 3.

(**A**) $S_N = 2k$, for $k = 3, 4, 5, \ldots$
(**B**) $S_N = 6k$, for $k = 1, 2, 3, \ldots$
(**C**) $S_N = 6k + 1$, for $k = 1, 2, 3, \ldots$
(**D**) $S_N = 6k - 1$, for $k = 1, 2, 3, \ldots$
(**E**) $S_N = 8k - 1$, for $k = 1, 2, 3, \ldots$

49. The diagram shows an equilateral triangle inscribed in a square. What is the ratio of the area of the triangle to the area of the square?



(**A**) $\frac{1}{2}$ (**B**) $\frac{\sqrt{3}}{4}$ (**C**) $2\sqrt{3} - 3$ (**D**) $\frac{3 - \sqrt{3}}{2}$ (**E**) $\frac{\sqrt{4 + 2\sqrt{3}}}{4}$

50. 2002 is a palindrome, meaning that the number is unchanged when the order of the digits is reversed. How many proper factors of 2002 are palindromes consisting of more than one digit?

(**A**) 1 (**B**) 2 (**C**) 3 (**D**) 4 (**E**) 5