2003 Comprehensive Test – Division I

1. Find the number of integer pairs (x, y) satisfying $|x| + |y| \leq 20$.

(**A**) 838 (**B**) 839 (**C**) 840 (**D**) 841 (**E**) 842

2. How many integers n, $1 \leq n \leq 100$, are *not* divisible by 2 or 3?

(**A**) 32 (**B**) 33 (**C**) 34 (**D**) 35 (**E**) 36

3. How many consecutive zeros does 100! have at the end?

(**A**) 10 (**B**) 20 (**C**) 24 (**D**) 26 (**E**) 30

4. Which is **not** a factor of $a^8 - b^8$?

(**A**) $a - b$ (**B**) $a + b$ (**C**) $a^2 + b^2$ (**D**) $a^3 + b^3$ (**E**) $a^4 + a^4$

5. Which of the following is equivalent to $-7 \leq 2 - 3x < 23$?

(**A**) $-7 < x \leq 3$ (**B**) $-7 \leq x < 3$ (**C**) $-3 < x \leq 7$
(**D**) $3 < x \leq 7$ (**E**) $3 \leq x < 7$

6. If $A : B : C = 3 : 4 : 5$, then $\dfrac{-3A + 2B + 5C}{A + B + C} = (?)$

(**A**) 1 (**B**) 2 (**C**) 3 (**D**) 4 (**E**) 5

7. Define the operation $\circ$ by $a \circ b = \dfrac{a^b - b^a}{a^b + b^a}$. If $3 \circ 4 = \dfrac{p}{q}$, where $\dfrac{p}{q}$ is a reduced fraction, then $q - p = (?)$

(**A**) 81 (**B**) 91 (**C**) 118 (**D**) 128 (**E**) 138

8. What is the sum of the coefficients of all the terms of $(2x - 3y)^7$ when it is expanded?

(**A**) -243 (**B**) -1 (**C**) 0 (**D**) 1 (**E**) 32

9. If $f\left(\dfrac{x+1}{x-1}\right) = 2x$, then $f(3) = (?)$

(**A**) 1 (**B**) 2 (**C**) 3 (**D**) 4 (**E**) 5

10. Solve $2^{16^x} = 16^{2^x}$ for x.

(**A**) $\dfrac{1}{2}$ (**B**) $\dfrac{2}{3}$ (**C**) $\dfrac{3}{4}$ (**D**) $\dfrac{4}{5}$ (**E**) $\dfrac{5}{6}$

11. If the equation $x^2 - ax + b = 0$ has two real solutions, $x = p$ and $x = q$, and $p \geq q$, then $p - q = (?)$

(**A**) $a^2 - 4b$ (**B**) $a^2 - 2b$ (**C**) $\sqrt{a^2 - 4b}$ (**D**) $\sqrt{a^2 + 4b}$ (**E**) $\sqrt{a^2 - 2b}$

12. If $i = \sqrt{-1}$, what is the value of i^{2003}?

(**A**) 0 (**B**) 1 (**C**) -1 (**D**) i (**E**) $-i$

13. Two students attempted to solve a quadratic equation in the form $x^2 + bx + c$. Although both students did their work correctly, the first miscopied the coefficient b and obtained the solution set $\{2, -2\}$. The second miscopied the constant term c and obtained the solution set $\{-1, -2\}$. What is the correct solution set?

(**A**) $\{-4, 1\}$ (**B**) $\{-2, 1\}$ (**C**) $\{-1, 1\}$ (**D**) $\{-1, 4\}$ (**E**) $\{1, 2\}$

14. If the height of a triangle $\triangle ABC$ is increased by 10%, and the width of $\triangle ABC$ is decreased by 10%, what will happen to the area of $\triangle ABC$?

(**A**) Same as the area of $\triangle ABC$.
(**B**) Increases by 10%.
(**C**) Decreases by 10%.
(**D**) Increases by 1%.
(**E**) Decreases by 1%.

15. A square and an equilateral triangle have equal areas. What is the ratio of the perimeter of the square to the perimeter of the triangle?

(**A**) $\frac{\sqrt[4]{3}}{2}$ (**B**) $\frac{\sqrt{3}}{2}$ (**C**) $\frac{\sqrt{3}}{4}$ (**D**) $\sqrt{\frac{2\sqrt{3}}{3}}$ (**E**) $\frac{\sqrt{4\sqrt{3}}}{3}$

16. If $x - 2$ is a factor of $x^4 - 3ax^2 + (5a - 2)x - 16$, then $a = (?)$

(**A**) -2 (**B**) -1 (**C**) 1 (**D**) 2 (**E**) 3

17. A candy jar now contains nine candies. After it was first filled, Monica came by and ate 60% of the candies. Then Ben came by and ate 25% of the remaining amount. How many candies were originally in the candy jar?

(**A**) 20 (**B**) 30 (**C**) 40 (**D**) 60 (**E**) 90

18. What is the distance from the center of the circle $x^2 + y^2 - 8x + 2y = 0$ to the vertex of the parabola $y = x^2 - 4x + 10$?

(**A**) 6 (**B**) 7 (**C**) $2\sqrt{13}$ (**D**) $\sqrt{53}$ (**E**) $\sqrt{71}$

19. Find $\sum_{n=1}^{5} n \cdot n!$.

(**A**) 719 (**B**) 720 (**C**) 730 (**D**) 739 (**E**) 751

20. If $0 < \theta < \frac{\pi}{2}$ and $\tan\theta = 3$, then $\sin 2\theta = (?)$

(**A**) $\frac{3}{10}$ (**B**) $\frac{3}{5}$ (**C**) $\frac{3\sqrt{10}}{10}$ (**D**) $\frac{3\sqrt{5}}{5}$ (**E**) $\frac{3\sqrt{10}}{5}$

21. The line perpendicular to $3x + 2y = 5$ at $(1, 1)$ can be written in the form $Ax + By = C$, where A, B, and C are integers and $\gcd(|A|, |B|) = 1$. What is the value of $|A + B + C|$?

(**A**) -1 (**B**) 1 (**C**) 2 (**D**) 3.5 (**E**) 4

22. If ω is a complex number satisfying the equation $\omega^2 + \omega + 1 = 0$, then $\omega^{12} = (?)$

(**A**) 1 (**B**) -1 (**C**) $-i$ (**D**) 16 (**E**) $16i$

23. What is the sum of the solutions to the equation $\log_2 x + \log_2(x+6) = 4$?

(**A**) -6 (**B**) -2 (**C**) 2 (**D**) 5 (**E**) 6

24. $\tan\left(\sin^{-1}\left(\frac{1}{2}\right)\right) = (?)$

(**A**) 0 (**B**) $\frac{\sqrt{3}}{3}$ (**C**) 1 (**D**) $\sqrt{3}$ (**E**) 2

25. What is the domain of the function $f(x) = \log x + \sqrt{7-3x}$?

(**A**) $-\infty < x < 0$ (**B**) $-\infty < x \leq \frac{7}{3}$ (**C**) $0 < x < \infty$

(**D**) $0 < x < \frac{7}{3}$ (**E**) $0 < x \leq \frac{7}{3}$

26. In $\triangle ABC$, we have $AB = 3$, $BC = 4$ and $CA = 5$. If M is the midpoint of $\overline{BC}$, what is the length of $\overline{AM}$?

(**A**) $\sqrt{13}$ (**B**) 4 (**C**) $\sqrt{19}$ (**D**) $\sqrt{21}$ (**E**) $2+2\sqrt{3}$

27. How many distinguishable 6-letter words can you spell using all six letters of TOMATO? (The "words" do not have to make sense.)

(**A**) 10 (**B**) 30 (**C**) 90 (**D**) 180 (**E**) 360

28. If $a^2+b^2+c^2-ab-ac-bc = 0$ and $a \neq 0$, then $\frac{ab+2ac+3bc}{a^2+b^2+c^2} = (?)$

(**A**) 0 (**B**) 1 (**C**) 2 (**D**) 3 (**E**) 4

29. Find the number of sets $\{a, b, c\}$ of three distinct positive integers, a, b, and c, that satisfy the equation $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$.

(**A**) 0 (**B**) 1 (**C**) 2 (**D**) 3 (**E**) 4

30. A cube is inscribed in a ball. What is the ratio of the volume of the ball to the volume of the cube?

(**A**) $\frac{\pi}{6}$ (**B**) $\frac{\pi}{3}$ (**C**) $\frac{\pi\sqrt{3}}{2}$ (**D**) $\frac{4\pi}{3}$ (**E**) $\frac{2\pi\sqrt{3}}{3}$

31. Find $\sum_{k=1}^{99} \frac{1}{\sqrt{k+1}+\sqrt{k}}$.

(**A**) 5 (**B**) 6 (**C**) 7 (**D**) 8 (**E**) 9

32. If $x - y = 3$ and $x^2 - y^2 = 9$, then $xy = (?)$

(**A**) 0 (**B**) 1 (**C**) 2 (**D**) 3 (**E**) 4

33. If the point (a, b) is the closest point on the curve $y = \sqrt{x}$ to the point $(1, 0)$, then $ab = (?)$

(**A**) $\sqrt{2}$ (**B**) 2 (**C**) $\frac{\sqrt{2}}{2}$ (**D**) $\frac{1}{2}$ (**E**) $\frac{\sqrt{2}}{4}$

34. How many elements are in the solution set of the equation $x^2 - 3|x| + 2 = 0$?

(**A**) 0 (**B**) 1 (**C**) 2 (**D**) 3 (**E**) 4

35. Find the sum of digits in all 3-digit integers whose digits are 1, 2, 3, 4, or 5, with no repetitions allowed.

(**A**) 270 (**B**) 450 (**C**) 540 (**D**) 630 (**E**) 720

36. Let $f(x)$ be a function with the following conditions: (i) $f(x) > 0$
(ii) $f(2) = 1$
(iii) $2f(x + y) = f(x)f(y)$
Find $f(1)$.

(**A**) $\sqrt{2}$ (**B**) $\sqrt[4]{2}$ (**C**) $\frac{\sqrt{2}}{2}$ (**D**) 1 (**E**) $\frac{1}{2}$

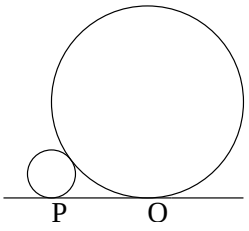
37. Two concentric circles have radii 4 and 6. What is the area of the annulus formed by the circles?

(**A**) 2π (**B**) 4π (**C**) 10π (**D**) 20π (**E**) 30π

38. Which one of the following is equivalent to $2^{\log_3 5}$?

(**A**) $2^{\log_5 3}$ (**B**) $5^{\log_3 2}$ (**C**) $3^{\log_2 3}$ (**D**) $5^{\log_2 3}$ (**E**) $3^{\log_5 2}$

39. The diagram shows circles with radii of 1 and 4. The circles are tangent to each other and $\overleftrightarrow{PQ}$ is tangent to both circles. Find PQ.



(**A**) 3 (**B**) $2\sqrt{2}$ (**C**) 4 (**D**) $\sqrt{17}$ (**E**) $\frac{14}{3}$

40. Solve the equation $\log_{\sin x} \cos x + \log_{\cos x} \tan x = 1$, where $0° < x < 90°$.

(**A**) $15°$ (**B**) $30°$ (**C**) $45°$ (**D**) $60°$ (**E**) $75°$

41. Simplify $\dfrac{\tan\theta}{\sec\theta - 1} + \dfrac{\tan\theta}{\sec\theta + 1}$.

(**A**) 1 (**B**) $2\sin\theta$ (**C**) $2\csc\theta$ (**D**) $2\cos\theta$ (**E**) $2\sec\theta$

42. If the parabola $y = x^2 + bx + c$ passes through $(-1, -1)$ and $(2, 14)$, what is the value of $b + c$?

(**A**) 6 (**B**) 7 (**C**) 8 (**D**) 9 (**E**) 10

43. A is the point $(-1, 0)$. The line $y = x - 1$ intersects the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{3} = 1$ at two points B and C. Find the perimeter of the triangle ABC.

(**A**) 4 (**B**) 5 (**C**) 6 (**D**) 7 (**E**) 8

44. If $x = 1 + \sqrt{1 + \sqrt{1 + \sqrt{\ldots}}}$, what is the value of $x + \dfrac{1}{x}$?

(**A**) 1 (**B**) 2 (**C**) 3 (**D**) 4 (**E**) 5

45. Let $f(x) = \dfrac{a^x + a^{-x}}{a^x - a^{-x}}$, where $a > 0$ and $a \neq 1$. If $f(p) = 2$, then $f(2p) = (?)$

(**A**) 3 (**B**) 4 (**C**) $\sqrt{3}$ (**D**) $\dfrac{5}{4}$ (**E**) $\dfrac{8}{7}$

46. Let $i = \sqrt{-1}$. Simplify $\left(\dfrac{1+i}{1-i}\right)^6 + \left(\dfrac{1-i}{1+i}\right)^6$.

(**A**) -2 (**B**) -1 (**C**) 0 (**D**) 1 (**E**) 2

47. If $\sin^2 x - 3\sin x \cos x - 4\cos^2 x = 0$ and $0° < x < 90°$, then $\tan x = (?)$

(**A**) $\dfrac{\sqrt{3}}{3}$ (**B**) 1 (**C**) $\sqrt{3}$ (**D**) 2 (**E**) 4

48. How many solutions does the equation $|x + 4| + |x - 2| = 6$ have?

(**A**) 0 (**B**) 1 (**C**) 2

(**D**) 3 (**E**) infinitely many

49. How many integers does the solution set of $\dfrac{x-3}{x+1} \leq 0$ contain?

(**A**) 2 (**B**) 4 (**C**) 6

(**D**) 8 (**E**) infinitely many

50. A function f takes a domain D *onto* a range R if, for each $y \in R$, there is some $x \in D$ for which $f(x) = y$. How many functions can be defined from the domain $D = \{1, 2, 3\}$ onto the range $R = \{4, 5\}$?

(**A**) 5 (**B**) 6 (**C**) 7 (**D**) 8 (**E**) 9