2003 Comprehensive Test – Division III

1. Solve $4^{x-3} = 8^{4-x}$.

(**A**) $-\frac{3}{2}$ (**B**) 1 (**C**) $\frac{12}{5}$ (**D**) $\frac{18}{5}$ $\star$ (**E**) 4

2. How many integers n, $1 \leq n \leq 100$, are *not* divisible by 2 or 3?

(**A**) 32 (**B**) 33 $\star$ (**C**) 34 (**D**) 35 (**E**) 36

3. $\left(\frac{9}{16}\right)^{3/2} = (?)$

(**A**) $\frac{27}{32}$ (**B**) $\frac{9}{24}$ (**C**) $\frac{27}{64}$ $\star$ (**D**) $\frac{3}{4}$ (**E**) $\frac{64}{27}$

4. Which is **not** a factor of $a^8 - b^8$?

(**A**) $a - b$ (**B**) $a + b$ (**C**) $a^2 + b^2$ (**D**) $a^3 + b^3$ $\star$ (**E**) $a^4 + a^4$

5. Which of the following is equivalent to $-7 \leq 2 - 3x < 23$?

(**A**) $-7 < x \leq 3$ $\star$ (**B**) $-7 \leq x < 3$ (**C**) $-3 < x \leq 7$

(**D**) $3 < x \leq 7$ (**E**) $3 \leq x < 7$

6. If $A : B : C = 3 : 4 : 5$, then $\frac{-3A + 2B + 5C}{A + B + C} = (?)$

(**A**) 1 (**B**) 2 $\star$ (**C**) 3 (**D**) 4 (**E**) 5

7. Define the operation $\circ$ by $a \circ b = \frac{a^b - b^a}{a^b + b^a}$. If $3 \circ 4 = \frac{p}{q}$, where $\frac{p}{q}$ is a reduced fraction, then $q - p = (?)$

(**A**) 81 (**B**) 91 (**C**) 118 (**D**) 128 $\star$ (**E**) 138

8. Solve the equation $\sqrt{\sqrt{x} + 9} = 4$ for x.

(**A**) 49 $\star$ (**B**) 52 (**C**) 64 (**D**) 81 (**E**) $\frac{81}{2}$

9. The average of three numbers is 64. When a fourth number is added to the set, the average drops to 62. What is the fourth number?

(**A**) 50 (**B**) 52 (**C**) 54 (**D**) 56 $\star$ (**E**) 58

10. $\frac{\sqrt{5}}{2} + \frac{2}{\sqrt{5}} = (?)$

(**A**) $\frac{\sqrt{5}}{3}$ (**B**) $\frac{9\sqrt{5}}{10}$ $\star$ (**C**) $\frac{3\sqrt{5}}{4}$ (**D**) 1 (**E**) $\frac{\sqrt{5}}{5}$

11. If the equation $x^2 - ax + b = 0$ has two real solutions, $x = p$ and $x = q$, and $p \geq q$, then $p - q = (?)$

(**A**) $a^2 - 4b$ (**B**) $a^2 - 2b$ (**C**) $\sqrt{a^2 - 4b}$ $\star$ (**D**) $\sqrt{a^2 + 4b}$ (**E**) $\sqrt{a^2 - 2b}$

12. If $i = \sqrt{-1}$, what is the value of i^{2003}?

(**A**) 0 (**B**) 1 (**C**) -1 (**D**) i (**E**) $-i$ $\star$

13. If a, b, and c are three different positive numbers and $\frac{1}{a} = \frac{1}{b} + \frac{1}{c}$, then $b = (?)$

(**A**) $\frac{ac}{c-a}$ $\star$ (**B**) $\frac{ac}{a-b+c}$ (**C**) $\frac{a}{c}$ (**D**) $\frac{c}{a}$ (**E**) $\frac{a-b}{c-b}$

14. If the height of a triangle $\triangle ABC$ is increased by 10%, and the width of $\triangle ABC$ is decreased by 10%, what will happen to the area of $\triangle ABC$?

(**A**) Same as the area of $\triangle ABC$.
(**B**) Increases by 10%.
(**C**) Decreases by 10%.
(**D**) Increases by 1%.
(**E**) Decreases by 1%. $\star$

15. A square and an equilateral triangle have equal areas. What is the ratio of the perimeter of the square to the perimeter of the triangle?

(**A**) $\frac{\sqrt[4]{3}}{2}$ (**B**) $\frac{\sqrt{3}}{2}$ (**C**) $\frac{\sqrt{3}}{4}$ (**D**) $\sqrt{\frac{2\sqrt{3}}{3}}$ (**E**) $\frac{\sqrt{4\sqrt{3}}}{3}$ $\star$

16. If $x - 2$ is a factor of $x^4 - 3ax^2 + (5a - 2)x - 16$, then $a = (?)$

(**A**) -2 $\star$ (**B**) -1 (**C**) 1 (**D**) 2 (**E**) 3

17. A candy jar now contains nine candies. After it was first filled, Monica came by and ate 60% of the candies. Then Ben came by and ate 25% of the remaining amount. How many candies were originally in the candy jar?

(**A**) 20 (**B**) 30 $\star$ (**C**) 40 (**D**) 60 (**E**) 90

18. What is the distance from the center of the circle $x^2 + y^2 - 8x + 2y = 0$ to the vertex of the parabola $y = x^2 - 4x + 10$?

(**A**) 6 (**B**) 7 (**C**) $2\sqrt{13}$ (**D**) $\sqrt{53}$ $\star$ (**E**) $\sqrt{71}$

19. Solve the equation $\frac{9}{4x} - \frac{3}{x^2} = \frac{2}{3x^2}$.

(**A**) -1 (**B**) 0 (**C**) $\frac{5}{27}$ (**D**) $\frac{3}{2}$ (**E**) $\frac{44}{27}$ $\star$

20. If $0 < \theta < \frac{\pi}{2}$ and $\tan\theta = 3$, then $\sin 2\theta = (?)$

(**A**) $\frac{3}{10}$ (**B**) $\frac{3}{5}$ $\star$ (**C**) $\frac{3\sqrt{10}}{10}$ (**D**) $\frac{3\sqrt{5}}{5}$ (**E**) $\frac{3\sqrt{10}}{5}$

21. If the lines $3x + 4y = 5$ and $-4x + By = 8$ are perpendicular, then $B = (?)$

(**A**) -2 (**B**) -1 (**C**) 2 (**D**) 3 $\star$ (**E**) $\frac{4}{3}$

22. If ω is a complex number satisfying the equation $\omega^2 + \omega + 1 = 0$, then $\omega^{12} = (?)$

(**A**) 1 $\star$ (**B**) -1 (**C**) $-i$ (**D**) 16 (**E**) $16i$

23. What is the sum of the solutions to the equation $\log_2 x + \log_2(x+6) = 4$?

(**A**) -6 (**B**) -2 (**C**) 2 $\star$ (**D**) 5 (**E**) 6

24. $\tan\left(\sin^{-1}\left(\frac{1}{2}\right)\right) = (?)$

(**A**) 0 (**B**) $\frac{\sqrt{3}}{3}$ $\star$ (**C**) 1 (**D**) $\sqrt{3}$ (**E**) 2

25. Write $\left(\frac{xy^3z^{-4}}{x^{-3}y^{-2}z^2}\right)^{-3}$ using only positive exponents.

(**A**) $\frac{x^{19}}{y^3z^8}$ (**B**) $\frac{x^8}{y^3z^8}$ (**C**) $\frac{z^6}{x^6y^6}$ (**D**) $\frac{z^4}{x^{12}y^{15}}$ (**E**) $\frac{z^{18}}{x^{12}y^{15}}$ $\star$

26. In $\triangle ABC$, we have $AB = 3$, $BC = 4$ and $CA = 5$. If M is the midpoint of $\overline{BC}$, what is the length of $\overline{AM}$?

(**A**) $\sqrt{13}$ $\star$ (**B**) 4 (**C**) $\sqrt{19}$ (**D**) $\sqrt{21}$ (**E**) $2+2\sqrt{3}$

27. How many distinguishable 6-letter words can you spell using all six letters of TOMATO? (The "words" do not have to make sense.)

(**A**) 10 (**B**) 30 (**C**) 90 (**D**) 180 $\star$ (**E**) 360

28. $\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \frac{3}{100000} + \frac{3}{1000000} + \ldots = (?)$

(**A**) $\frac{3}{7}$ (**B**) $\frac{1}{3}$ $\star$ (**C**) $\frac{1}{2}$ (**D**) 1 (**E**) $\frac{4}{3}$

29. If $f(x) = \frac{2x}{3x+5}$ and $g(x) = \frac{3x+1}{4x}$, then $g(f(2)) = (?)$

(**A**) 1 (**B**) $\frac{23}{16}$ $\star$ (**C**) 2 (**D**) $\frac{5}{11}$ (**E**) 3

30. A cube is inscribed in a ball. What is the ratio of the volume of the ball to the volume of the cube?

(**A**) $\frac{\pi}{6}$ (**B**) $\frac{\pi}{3}$ (**C**) $\frac{\pi\sqrt{3}}{2}$ $\star$ (**D**) $\frac{4\pi}{3}$ (**E**) $\frac{2\pi\sqrt{3}}{3}$

31. Solve the inequality $1 < 3x + 4 \leq 13$ for x.

(**A**) $(0, 5]$ (**B**) $(0, 5)$ (**C**) $(1, 3]$ (**D**) $[-1, 3]$ (**E**) $(-1, 3]$ $\star$

32. If $x - y = 3$ and $x^2 - y^2 = 9$, then $xy = (?)$

(**A**) 0 $\star$ (**B**) 1 (**C**) 2 (**D**) 3 (**E**) 4

33. If the point (a, b) is the closest point on the curve $y = \sqrt{x}$ to the point $(1, 0)$, then $ab = (?)$

(**A**) $\sqrt{2}$ (**B**) 2 (**C**) $\frac{\sqrt{2}}{2}$ (**D**) $\frac{1}{2}$ (**E**) $\frac{\sqrt{2}}{4}$ $\star$

34. How many elements are in the solution set of the equation $x^2 - 3|x| + 2 = 0$?

(**A**) 0 (**B**) 1 (**C**) 2 (**D**) 3 (**E**) 4 $\star$

35. What is the midpoint of the segment whose endpoints are $(-1, 9)$ and $(-13, 23)$?

(**A**) $(-4, 16)$ (**B**) $(-7, 16)$ $\star$ (**C**) $(7, 16)$ (**D**) $(-5, 12)$ (**E**) $(-7, 14)$

36. What is the remainder when $4x^2 - 8x + 11$ is divided by $2x - 3$?

(**A**) 4 (**B**) $\frac{5}{2}$ (**C**) $\frac{8}{3}$ (**D**) 8 $\star$ (**E**) $\frac{15}{2}$

37. Two concentric circles have radii 4 and 6. What is the area of the annulus formed by the circles?

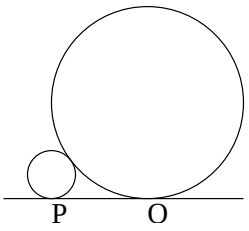
(**A**) 2π (**B**) 4π (**C**) 10π (**D**) 20π $\star$ (**E**) 30π

38. $\sin\left(\frac{\pi}{3} + \frac{\pi}{4}\right) = (?)$

(**A**) $\frac{\sqrt{2}}{4}(\sqrt{3} - 1)$ (**B**) $\frac{\sqrt{2}}{4}(\sqrt{3} + 1)$ $\star$ (**C**) $\frac{\sqrt{2} + \sqrt{3}}{4}$

(**D**) $\frac{\sqrt{3}}{4}(\sqrt{2} - 1)$ (**E**) $\frac{\sqrt{3}}{4}(\sqrt{2} + 1$

39. The diagram shows circles with radii of 1 and 4. The circles are tangent to each other and $\overleftrightarrow{PQ}$ is tangent to both circles. Find PQ.



(**A**) 3 (**B**) $2\sqrt{2}$ (**C**) 4 $\star$ (**D**) $\sqrt{17}$ (**E**) $\frac{14}{3}$

40. Solve $\log(2x + 3) = \log 11 + \log 3$.

(**A**) 5 (**B**) 8 (**C**) 15 $\star$ (**D**) $\frac{33}{2}$ (**E**) 33

41. Simplify $\dfrac{\tan\theta}{\sec\theta - 1} + \dfrac{\tan\theta}{\sec\theta + 1}$.

(**A**) 1 (**B**) $2\sin\theta$ (**C**) $2\csc\theta$ $\star$ (**D**) $2\cos\theta$ (**E**) $2\sec\theta$

42. Find the determinant of the matrix $\begin{bmatrix} 0 & 0 & 0 & 3 \\ 0 & 6 & 3 & -4 \\ 0 & 0 & 4 & 2 \\ 1 & 0 & -2 & 10 \end{bmatrix}$.

(**A**) -72 $\star$ (**B**) -50 (**C**) 0 (**D**) 72 (**E**) 113

43. If a and b are the solutions of $(x - 6)(x - 2) = 21$, then $|a - b| = (?)$

(**A**) 4 (**B**) 5 (**C**) 8 (**D**) 10 $\star$ (**E**) 12

44. Find the center of the circle $x^2 + y^2 + 4x - 2y + 1 = 0$.

(**A**) $(1, -2)$ (**B**) $(-1, 2)$ (**C**) $(-2, 1)$ $\star$ (**D**) $(-2, 2)$ (**E**) $(2, -2)$

45. Let $f(x) = \dfrac{a^x + a^{-x}}{a^x - a^{-x}}$, where $a > 0$ and $a \neq 1$. If $f(p) = 2$, then $f(2p) = (?)$

(**A**) 3 (**B**) 4 (**C**) $\sqrt{3}$ (**D**) $\frac{5}{4}$ $\star$ (**E**) $\frac{8}{7}$

46. Let $i = \sqrt{-1}$. Simplify $\left(\frac{1+i}{1-i}\right)^6 + \left(\frac{1-i}{1+i}\right)^6$.

(**A**) -2 ⋆ (**B**) -1 (**C**) 0 (**D**) 1 (**E**) 2

47. If $\sin^2 x - 3\sin x \cos x - 4\cos^2 x = 0$ and $0° < x < 90°$, then $\tan x = (?)$

(**A**) $\frac{\sqrt{3}}{3}$ (**B**) 1 (**C**) $\sqrt{3}$ (**D**) 2 (**E**) 4 ⋆

48. How many solutions does the equation $|x+4| + |x-2| = 6$ have?

(**A**) 0 (**B**) 1 (**C**) 2

(**D**) 3 (**E**) infinitely many ⋆

49. How many integers does the solution set of $\frac{x-3}{x+1} \leq 0$ contain?

(**A**) 2 (**B**) 4 ⋆ (**C**) 6

(**D**) 8 (**E**) infinitely many

50. A function f takes a domain D *onto* a range R if, for each $y \in R$, there is some $x \in D$ for which $f(x) = y$. How many functions can be defined from the domain $D = \{1, 2, 3\}$ onto the range $R = \{4, 5\}$?

(**A**) 5 (**B**) 6 ⋆ (**C**) 7 (**D**) 8 (**E**) 9