ALGEBRA II WITH TRIGONOMETRY EXAMINATION
2001
ALABAMA STATE-WIDE MATHEMATICS CONTEST

Construction of this test directed
by
James Dupuy, Birmingham

INSTRUCTIONS

This test consists of 50 multiple choice questions. For each question, choose the best of the five answer choices labeled A, B, C, D, and E.

The test will be scored as follows: 5 points for each correct answer, 1 point for each question left unanswered, and 0 points for each wrong answer. (Thus a "perfect paper" with all questions answered correctly earns a score of 250, a blank paper earns a score of 50, and a paper with all questions answered incorrectly earns a score of 0.) Random guessing will not, on average, either increase or decrease your score. However, if you can eliminate one or more of the answer choices as wrong, then it is to your advantage to guess among the remaining choices.

The questions have not been arranged in order of difficulty.

All variables and constants represent real numbers, except when a particular problem indicates otherwise.

Diagrams are not necessarily to scale.

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1. Evaluate $8^{-2/3}$.

(A) $-\dfrac{16}{3}$ **(B)** -4 **(C)** $\dfrac{3}{16}$ **(D)** $\dfrac{1}{4}$ **(E)** $\dfrac{1}{8}$

2. One of the factors of $8x^6 - 7x^3 - 1$ is

(A) $4x^2 - 2x + 1$ **(B)** $x^2 - x + 1$ **(C)** $2x^2 - 4x + 1$ **(D)** $2x - 1$ **(E)** $x^2 + x - 1$

3. Rationalize the denominator and write your answer in lowest terms: $\dfrac{2\sqrt{6} + \sqrt{3}}{\sqrt{6} - \sqrt{3}}$

(A) $5 + 2\sqrt{2}$ **(B)** $4 + \sqrt{2}$ **(C)** $5 + 3\sqrt{2}$ **(D)** $3 + 6\sqrt{2}$ **(E)** $1 + \sqrt{2}$

4. Evaluate $\log_9 243$.

(A) 3 **(B)** $\frac{3}{2}$ **(C)** $\frac{5}{2}$ **(D)** $\frac{7}{2}$ **(E)** 27

5. What is the distance between the centers of the circles $x^2 + y^2 + 4x - 6y = 2$ and $x^2 + y^2 - 8x + 10y = 0$?

(A) 10 **(B)** 100 **(C)** $2\sqrt{2}$ **(D)** 8 **(E)** 14

6. Let $f(x) = e^x$. Then the quotient $\dfrac{f(x + 0.1) - f(x)}{0.1}$ is

(A) 1 **(B)** directly proportional to $f(x)$ **(C)** inversely proportional to $f(x)$
(D) independent of x **(E)** equal to $f(x)$

7. Find the real number a for which the lines $-4x+3y = 7$ and $ax-2y = 4$ will be perpendicular.

(A) $\dfrac{3}{2}$ **(B)** $\dfrac{2}{3}$ **(C)** $-\dfrac{3}{2}$ **(D)** $-\dfrac{4}{3}$ **(E)** $\dfrac{3}{4}$

8. Find the sum of the infinite series:

$$\frac{1}{7} + \frac{2}{7^2} + \frac{3}{7^3} + \frac{1}{7^4} + \frac{2}{7^5} + \frac{3}{7^6} + \dots$$

where the numerators repeat with period 1, 2, 3.

(A) $\dfrac{49}{57}$ **(B)** $\dfrac{33}{172}$ **(C)** $\dfrac{49}{171}$ **(D)** $\dfrac{11}{57}$ **(E)** $\dfrac{11}{49}$

9. Find all real numbers x that satisfy the equation $2[x]^2 + 5[x] - 3 = 0$ where $[x]$ is the largest integer less than or equal to x.

(A) $x = -3$ and $x = \frac{1}{2}$ **(B)** $x = -3$ and $x = 0$ **(C)** $[-3, -2)$ **(D)** $(-4, -3]$ **(E)** $[-3, 0)$

10. Find a point on the y-axis that is equidistant from $(1, 1)$ and $(-5, 5)$.

(A) $(3, -2)$ **(B)** $(0, -2)$ **(C)** $(0, -4)$ **(D)** $(0, 1)$ **(E)** $(0, -3)$

11. Consider the quadratic equation $x^2 + px + q = 0$, where $p \cdot q \neq 0$. If one root is three times the other root, find p^2/q.

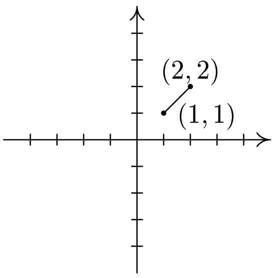
(A) $\dfrac{16}{3}$ **(B)** $-\dfrac{16}{3}$ **(C)** $-\dfrac{4}{3}$ **(D)** $\dfrac{16}{9}$ **(E)** $-\dfrac{4}{9}$

12. What is the domain of the function $g(x) = \sqrt{x^2 - 2x}$?

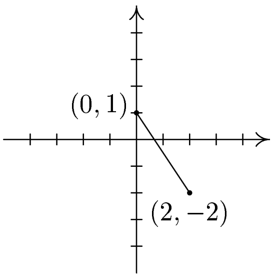
(A) $(-\infty, 0)$ or $[2, \infty)$ **(B)** $(-\infty, 0]$ or $[2, \infty)$ **(C)** $(0, 2]$ **(D)** $[0, 2]$ **(E)** All real numbers.

13.

The graph to your right is the graph of $y = f(x)$. Which of the graphs below is the graph of $y = -f(x-1) + 2$?



(A)



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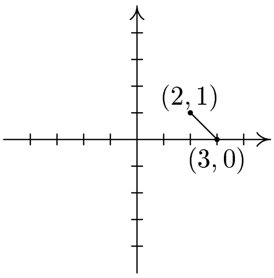

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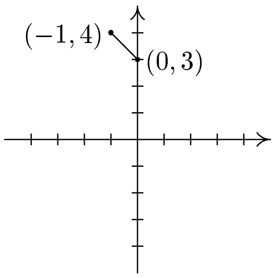

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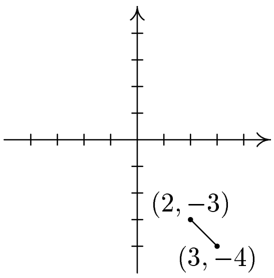

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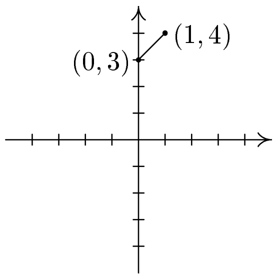


14. If $f(x) = \dfrac{4}{x-1}$ and $g(x) = 2x$ then the solution of $f(g(x)) = g(f(x))$ is $x =$

(A) 2 only **(B)** $\frac{1}{4}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{1}{2}$ **(E)** 2 or -1

15. A person has a hours at his disposal. How far (in miles) may he ride in a car traveling b miles per hour and yet have time to return on foot walking at c miles per hour?

(A) $\dfrac{b^2a}{b+a}$ **(B)** $\dfrac{ba-ac}{ab+c}$ **(C)** $\dfrac{abc}{b+c}$ **(D)** $\dfrac{ac}{b+c}$ **(E)** $\dfrac{b+c}{ac}$

16. Suppose z is a complex number of the form $z = a + bi$ where a and b are real numbers and $i = \sqrt{-1}$, and $\bar{z}$ denotes the conjugate of z. Of the following, which one is not necessarily real?

(A) $z + \bar{z}$ **(B)** $z \cdot \bar{z}$ **(C)** $\dfrac{z-\bar{z}}{2i}$ **(D)** $|z|$ **(E)** z^2

17. Let $f(x)$ be a linear function with $f(1) = 2$ and $f(3) = 8$. Find $f(5)$.

(A) $\frac{10}{3}$ **(B)** 20 **(C)** 16 **(D)** 10 **(E)** 14

18. Let a be a real number and consider the graph of $\dfrac{x^2}{7-a} + \dfrac{y^2}{4-a} = 1$. For what values of a is the graph a hyperbola?

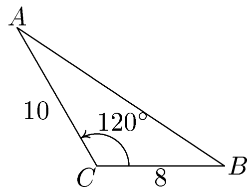
(A) $(-\infty, 4)$ **(B)** $(4, 7)$ **(C)** $(7, \infty)$ **(D)** $(-\infty, 4)$ or $(7, \infty)$
(E) All real numbers except $a = 4$ and $a = 7$

19. If $x = \sec\theta$ and $0 \le \theta < \frac{\pi}{2}$, express $\dfrac{\sqrt{x^2-1}}{x}$ as a trigonometric function in simplest form.

(A) $\sin\theta$ **(B)** $\cos\theta$ **(C)** $\tan\theta$ **(D)** $\dfrac{\cos\theta}{\sin^2\theta}$ **(E)** $\csc\theta$

20.

In the triangle to your right, $BC = 8$, $CA = 10$, and $\angle C = 120°$. Find the length of AB.



(A) 244 **(B)** $2\sqrt{61}$ **(C)** 84 **(D)** $2\sqrt{41}$ **(E)** $2\sqrt{31}$

21. What is the smallest value of x satisfying the determinant equation $\begin{vmatrix} x & 2 & 5 \\ 1 & 0 & 4 \\ x^2 & -1 & 2 \end{vmatrix} = 3$?

(A) -1 **(B)** 1 **(C)** -3 **(D)** $-\frac{1}{2}$ **(E)** $-\frac{3}{2}$

22. Let $\log_{10} 2 = a$ and $\log_{10} 3 = b$. Find $\log_{10} \dfrac{64}{15}$ in terms of a and b.

(A) $5a - b - 1$ **(B)** $5a - b + 1$ **(C)** $7a - b + 1$ **(D)** $7a - b - 1$ **(E)** $7a + b - 1$

23. Suppose that y varies inversely as x^2 and z varies directly as x^2. When $x = 2$, $y + z = 340$; when $x = 1$, $y - z = 1275$. For what values of x is $y = z$?

(A) ± 12 **(B)** ± 16 **(C)** ± 4 **(D)** ± 2 **(E)** ± 8

24. Four numbers are in geometric progression. The sum of the first two is 44 and of the last two 396. If the common ratio is negative, find the first term of the progression.

(A) 11 **(B)** -22 **(C)** -11 **(D)** 22 **(E)** -21

25. Find the area of the region bounded by the graphs of $y = |x| + 1$ and $y = 5$.

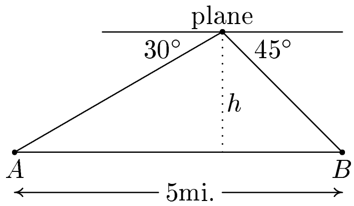
(A) 8 **(B)** 12 **(C)** 16 **(D)** 24 **(E)** 32

26. Find the absolute value of $\dfrac{2+3i}{1+i}$, where $i = \sqrt{-1}$.

(A) $\dfrac{5}{2}$ **(B)** $\dfrac{13}{2}$ **(C)** $\dfrac{\sqrt{26}}{2}$ **(D)** $\dfrac{13\sqrt{2}}{2}$ **(E)** $\dfrac{\sqrt{13}}{2}$

27.

A pilot is flying over a straight highway. He determines the angles of depression to two mileposts 5 miles apart to be 30° and 45°, as shown in the figure. Find the elevation h of the plane, in miles.



(A) $\frac{5\sqrt{2}}{2}$ **(B)** $\frac{5(3+\sqrt{3})}{2}$ **(C)** $\frac{5(\sqrt{3}+1)}{2}$ **(D)** $\frac{5(\sqrt{3}-1)}{2}$ **(E)** $\frac{5(3-\sqrt{3})}{2}$

28. If the difference between the roots of $x^2 + px + q = 0$ is the same as the difference between the roots of $x^2 + qx + p = 0$, and $p \neq q$, then what is $p + q$?

(A) 4 **(B)** -4 **(C)** 2 **(D)** -2 **(E)** 0

29. In the product $(x + a)(x + b)(x + c)$, the coefficient of x^2 is 0, and in the product $(x - a)(x + b)(x + c)$, the coefficient of x is 0; also, the coefficient of x in the first product is the same as the coefficient of x^2 in the second product. What is (are) the value(s) of a?

(A) 0 or 1 **(B)** 0 only **(C)** -1 or 2 **(D)** 1 only **(E)** -1 only

30. The sum of two numbers is S, and their difference is $\frac{1}{m}^{\text{th}}$ of the larger number. What is the larger number?

(A) $\frac{mS}{2m+1}$ **(B)** $\frac{mS+1}{2m}$ **(C)** $\frac{S}{m(2m+1)}$ **(D)** $\frac{S(m-1)}{2m-1}$ **(E)** $\frac{mS}{2m-1}$

31. If a committee of 6 members is to be chosen from among 5 Democrats and 3 Replublicans such that at least 2 members of each party serve on the committee, how many committees are possible?

(A) 40 **(B)** 30 **(C)** 25 **(D)** 20 **(E)** 35

32. Solve $\log_2 x + \log_4 x + \log_8 x = 11$ for x.

(A) 16 **(B)** 32 **(C)** 48 **(D)** 64 **(E)** 128

33. Solve $2\sin\theta\cos\theta + \cos\theta = 0$ for θ where $0 \leq \theta < 2\pi$.

(A) $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{4\pi}{3}, \frac{5\pi}{3}$ **(B)** $0, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}$ **(C)** $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$ **(D)** $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{11\pi}{6}, \frac{11\pi}{6}$ **(E)** $0, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$

34. If the expression $x^3y^4z^2$ were written without exponents, in how many ways could the letters be arranged?

(A) 24 **(B)** 288 **(C)** 2520 **(D)** 1260 **(E)** $9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

35. What is the maximum value of $g(x) = 4 - 2x - x^2$

(A) -1 **(B)** 1 **(C)** 5 **(D)** 7 **(E)** 3

36. The number $(3^{-1} + 2^{-1})^{-2}$ is equivalent to

(A) 13 **(B)** 25 **(C)** $\frac{25}{35}$ **(D)** $\frac{36}{25}$ **(E)** $\frac{25}{4}$

37. Where defined, $\dfrac{x^2y^{-1} - y^2x^{-1}}{x^{-2} + x^{-1}y^{-1} + y^{-2}}$ is equivalent to

(A) $\dfrac{x^2 - y^2}{xy(x^2 + xy + y^2)}$ **(B)** $\dfrac{xy(x-y)^3}{x^2 + xy + y^2}$ **(C)** $x - y$ **(D)** $xy(x - y)$
(E) $\dfrac{x - y}{xy(x^2 + xy + y^2)}$

38. How many solutions are there to the equation $t^2 = |3t - 2|$?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

39. Let $w = \dfrac{-1 + i\sqrt{3}}{2}$ where $i = \sqrt{-1}$. Compute the product $(1 - w + w^2)(1 + w + w^2)$.

(A) 1 **(B)** -3 **(C)** 0 **(D)** 13 **(E)** 4

40. Solve the inequality $\dfrac{x+3}{x-1} \leq 0$.

(A) $[-3, 1)$ **(B)** $(-\infty, -3)$ **(C)** $[-3, -1]$ **(D)** $(-\infty, -3]$ or $[1, \infty)$
(E) $(-\infty, -3]$ or $(1, \infty)$

41. Let $A = \begin{pmatrix} 3 & 2 & -3 \\ 5 & 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ 4 & -1 \\ 3 & -2 \end{pmatrix}$. Find the determinant of BA.

(A) 4 **(B)** 0 **(C)** -56 **(D)** -38 **(E)** 220

42. The perimeter of a rectangle is p and its area is a. The longest edge length of the rectangle in terms of p and a is

(A) $\dfrac{p + \sqrt{p^2 - 4a}}{4}$ **(B)** $\dfrac{p - \sqrt{p^2 - 4a}}{4}$ **(C)** $\dfrac{p + \sqrt{p^2 - 16a}}{4}$ **(D)** $\dfrac{p - \sqrt{p^2 - 16a}}{4}$
(E) $\dfrac{p + \sqrt{p^2 - 8a}}{4}$

43. In a class of 100, 40 students study Spanish, 35 study German, and 27 study French. Two study all 3 subjects, while 3 study Spanish and German only and 40 do not study either Spanish or French. Finally, 20 students study no language at all. How many students who study German do not study Spanish?

(A) 25 **(B)** 30 **(C)** 31 **(D)** 33 **(E)** 34

44. A door has the shape of an elliptical arch (half ellipse) that is 10 feet wide and 4 feet high at the center. A box two feet wide is to be pushed through the door. How high (in feet) can the box be?

(A) 4 **(B)** 3 **(C)** $\dfrac{4\sqrt{6}}{5}$ **(D)** $\dfrac{8\sqrt{6}}{5}$ **(E)** $\dfrac{2\sqrt{3}}{3}$

45. Find the product of the solutions of

$$x^{\log_{10} x} = \frac{1000}{x}$$

(A) 1,000,000 **(B)** 1000 **(C)** 100 **(D)** 1/10 **(E)** 1/1000

46. The number $3^{32} - 1$ is divisible by two integers between 80 and 90. What is the product of these two numbers?

(A) 6804 **(B)** 6560 **(C)** 7140 **(D)** 7308 **(E)** 7209

47. A four digit number is formed by randomly assigning a "2" or a "6" to each digit. What is the probability that the number is divisible by 11?

(A) 1/2 **(B)** 3/8 **(C)** 5/16 **(D)** 1/16 **(E)** 1/8

48. Find the number of ordered pairs (a, b) where a and b are integers, $|a| \leq 10$, $|b| \leq 10$, and a and b satisfy

$$ax + 2 < 3x + b \qquad \text{for all real } x < 0$$

(A) 7 **(B)** 8 **(C)** 64 **(D)** 71 **(E)** 72

49. How many zeros are at the end of 17! expressed in base 8?

(A) 17 **(B)** 9 **(C)** 5 **(D)** 3 **(E)** 2

50. For integers $n \geq 1$, let

$$I_n = 2 + 2 \cdot 2^2 + 3 \cdot 2^3 + 4 \cdot 2^4 + \cdots + n \cdot 2^n$$

The digits of I_{21} sum to

(A) 40 **(B)** 41 **(C)** 42 **(D)** 43 **(E)** 44