2006

ALABAMA

STATEWIDE MATHEMATICS CONTEST

First Round : February 18, 2006
Second Round: April 22, 2006 at The University of Alabama

DIVISION II COMPREHENSIVE EXAM

Construction of this test directed
by
Paul Allen, The University of Alabama

INSTRUCTIONS

This test consists of 50 multiple choice questions. The questions have not been arranged in order of difficulty. For each question, choose the best of the five answer choices labeled A, B, C, D, and E.

The test will be scored as follows: 5 points for each correct answer, 1 point for each question left unanswered, and 0 points for each wrong answer. (Thus a "perfect paper" with all questions answered correctly earns a score of 250, a blank paper earns a score of 50, and a paper with all questions answered incorrectly earns a score of 0.)

Random guessing will not, on average, either increase or decrease your score. However, if you can eliminate one of more of the answer choices as wrong, then it is to your advantage to guess among the remaining choices.

- All variables and constants, except those indicated otherwise, represent real numbers.
- Diagrams are not necessarily to scale.

We use the following geometric notation:

- If A and B are points, then:
  $\overline{AB}$ is the segment between A and B
  $\overleftrightarrow{AB}$ is the line containing A and B
  $\overrightarrow{AB}$ is the ray from A through B
  AB is the distance between A and B
- If A is an angle, then:
  $m\angle A$ is the measure of angle A in degrees
- If A and B are points on a circle, then:
  $\overset{\frown}{AB}$ is the arc between A and B
  $m\overset{\frown}{AB}$ is the measure of $\overset{\frown}{AB}$ in degrees

Editing by Zhijian Wu, The University of Alabama
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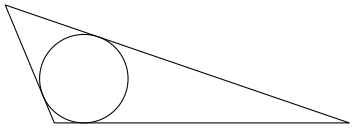
1. Find the sum of the following series: $\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\frac{5}{32}+\cdots$.

(**A**) $\frac{3}{2}$ (**B**) $\frac{\pi}{2}$ (**C**) $\boxed{2}$ (**D**) $\frac{2\pi}{3}$ (**E**) $\sqrt{5}$

2. What is the remainder when 2^{2006} is divided by 17?

(**A**) 0 (**B**) 1 (**C**) 5 (**D**) 11 (**E**) $\boxed{13}$

3. The triangle shown has sides of length 13, 30, and 37. What is the radius of the inscribed circle?



(**A**) $7+\sqrt{2}$ (**B**) $\boxed{\frac{9}{2}}$ (**C**) $7-\sqrt{2}$ (**D**) $\frac{7}{2}$ (**E**) None of these

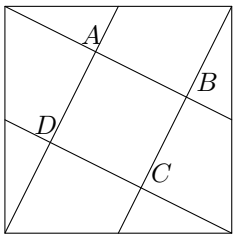
4. If $x^3=1$ and $x\neq 1$, what is $x(1+x)$?

(**A**) 1 (**B**) 2 (**C**) $\boxed{-1}$ (**D**) 0 (**E**) None of these

5. Which one of the following is the remainder when $x+x^7+x^{16}+x^{37}$ is divided by x^4-x?

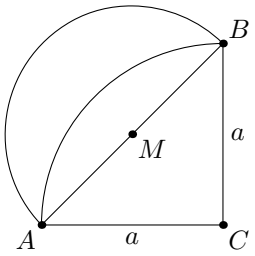
(**A**) 4 (**B**) $1+x^2$ (**C**) $\boxed{4x}$ (**D**) $x+x^3$ (**E**) $x+3x^2$

6. Lines from the vertices of a unit square are drawn to the midpoints of the sides as shown in the figure. Find the area of the quadrilateral $ABCD$.



(**A**) $\frac{\sqrt{2}}{9}$ (**B**) $\frac{1}{4}$ (**C**) $\frac{\sqrt{3}}{9}$ (**D**) $\frac{1}{\sqrt{8}}$ (**E**) $\boxed{\frac{1}{5}}$

7. A lune is a region bounded by two circular arcs of unequal radii as shown. Starting with the isosceles right Triangle ABC, draw a circular arc with center C and radius a. Let M be the midpoint of the line segment $\overline{AB}$. Then draw the circular arc with center M and radius AM. Which one of the following is the area of the lune?

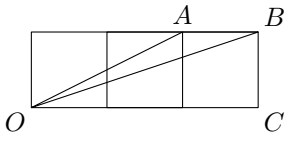


(**A**) $\frac{a^2}{8}$ (**B**) $\frac{\pi a^2}{4}$ (**C**) $\frac{a^2}{4}$ (**D**) $\frac{\sqrt{\pi}a^2}{4}$ (**E**) $\boxed{\frac{a^2}{2}}$

8. Find the polar form of the rectangular equation $x^2=8y$.

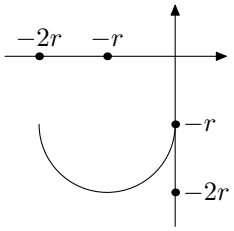
(**A**) $\boxed{r=8\sec\theta\tan\theta}$ (**B**) $r=8\cos\theta\sin\theta$ (**C**) $r^2=8\theta$ (**D**) $r=8\tan\theta$ (**E**) $r=8\cos\theta$

9. Three identical squares are placed side-by-side as indicated in the figure. Lines are then drawn from O to A and from O to B. Find the measure of angle AOC plus angle BOC.



(**A**) $\boxed{45^\circ}$ (**B**) 51.5° (**C**) 55° (**D**) 58.5° (**E**) 65°

10. The graph of $y = f(x)$ is the semicircular arc of radius r as shown. Which one of the following equations is the function $y = f(x)$?



(**A**) $y = -\sqrt{2rx - x^2} - r$ (**B**) $\boxed{y = -\sqrt{-2rx - x^2} - r}$

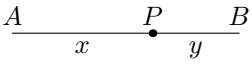
(**C**) $y = -\sqrt{2rx + x^2} - r$ (**D**) $y = -\sqrt{x^2 - 2rx} - r$ (**E**) None of these

11. Consider the polynomial equation $x^4 - 2x^3 + 3x^2 - 4x + 1 = 0$. Which one of the following statements describes correctly the solution set of this equation?

(**A**) four non-real complex zeros (**B**) four positive zeros

(**C**) two positive and two negative zeros (**D**) $\boxed{\text{no negative zeros}}$ (**E**) None of these

12. The line segment $\overline{AB}$ is divided into two unequal pieces by point P. Let x and y denote the length of the longer and shorter pieces as indicated in the picture. If $\dfrac{x+y}{x} = \dfrac{x}{y}$ find the ratio $\dfrac{y}{x}$.



(**A**) $\boxed{\dfrac{\sqrt{5}-1}{2}}$ (**B**) $\dfrac{2}{3}$ (**C**) $\dfrac{3-\sqrt{5}}{2}$ (**D**) $\dfrac{3}{5}$ (**E**) None of these

13. The number $\sqrt{18 + \sqrt{308}}$ can be written in the form $\sqrt{a} + \sqrt{b}$ where a and b are whole numbers and $a > b$. What is the value of $a - b$?

(**A**) 8 (**B**) 12 (**C**) 16 (**D**) 20 (**E**) $\boxed{\text{None of these}}$

14. Find the sum of the solutions of the equation: $\log x + \log(x + 30) = 3$.

(**A**) -30 (**B**) $\boxed{20}$ (**C**) -40 (**D**) 30 (**E**) There are no solutions

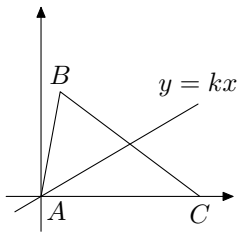
15. A pair of fair dice is cast. What is the probability that the sum of the numbers falling uppermost is 7 if it is known that one of the numbers is a 5?

(**A**) $\dfrac{2}{9}$ (**B**) $\dfrac{2}{35}$ (**C**) $\dfrac{1}{6}$ (**D**) $\dfrac{4}{35}$ (**E**) $\boxed{\dfrac{2}{11}}$

16. The equation $x^4 + 2x^3 - 3x^2 + 4x + 5 = 0$ has four zeros. What is their sum?

(**A**) -4 (**B**) 3 (**C**) $\boxed{-2}$ (**D**) 5 (**E**) -1

17. Triangle ABC has vertices $(0, 0)$, $(11, 60)$, and $(91, 0)$; respectively. The line $y = kx$ cuts the triangle into two triangles of equal area. Find k.



(**A**) $\dfrac{4}{3}$ (**B**) $\dfrac{25}{37}$ (**C**) $\dfrac{3}{5}$ (**D**) $\dfrac{4}{7}$ (**E**) $\boxed{\dfrac{30}{51}}$

18. If $g\left(\sqrt{\dfrac{x-1}{x+1}}\right) = 3x$, find $g(3)$.

(**A**) $\boxed{-\dfrac{15}{4}}$ (**B**) $\dfrac{\sqrt{2}}{2}$ (**C**) 9 (**D**) $\dfrac{\sqrt{3}}{3}$ (**E**) $-\dfrac{5}{4}$

19. Find the largest positive number δ such that $|\sqrt{x} - 2| < 0.1$ whenever $|x - 4| < \delta$.

(**A**) $\delta = 0.2$ (**B**) $\boxed{\delta = 0.39}$ (**C**) $\delta = 0.4$ (**D**) $\delta = 0.41$ (**E**) $\delta = 0.6$

20. Urn A contains 4 white balls and 6 black balls. Urn B contains 3 white balls and 5 black balls. A ball (the transferred ball) is drawn from urn A and then transferred to urn B. A ball (the second ball) is then drawn from urn B. What is the probability that the transferred ball was black given that the second ball drawn was write?

(**A**) $\dfrac{5}{9}$ (**B**) $\boxed{\dfrac{9}{17}}$ (**C**) $\dfrac{4}{9}$ (**D**) $\dfrac{6}{11}$ (**E**) $\dfrac{8}{45}$

21. Solve the given equation for x: $4^{2x+3} = \left(\dfrac{1}{8}\right)^{5-x}$.

(**A**) -11 (**B**) $\dfrac{15}{4}$ (**C**) -15 (**D**) $\dfrac{12}{5}$ (**E**) $\boxed{-21}$

22. All but two students in Mrs. Swift's math class took test three, and their calculated average was 73. The two students who missed the test later took a make-up exam and their grades were 60 and 100. Mrs. Swift the recalculated the class average for test three and determined the average to be 74. How many students are in Mrs. Swift's math class?

(**A**) 12 (**B**) $\boxed{14}$ (**C**) 16 (**D**) 18 (**E**) 20

23. Find the distance from the point $(1, -1)$ to the line $x + 2y = 4$.

(**A**) $\sqrt{2}$ (**B**) 3 (**C**) $\sqrt{3}$ (**D**) 4 (**E**) $\boxed{\sqrt{5}}$

24. A sphere is inscribed in a cube. What is the ratio of the surface area of the sphere to the surface of the cube?

(**A**) $\dfrac{2\pi}{3}$ (**B**) $\boxed{\dfrac{\pi}{6}}$ (**C**) $\dfrac{4\pi}{3}$ (**D**) $\dfrac{\pi}{12}$ (**E**) $\dfrac{8\pi}{3}$

25. If $(x^2 + y^2)^2 = 100$ and $(x^2 - y^2)^2 = 60$, then which one of the following is a possible value of xy?

(**A**) -6 (**B**) -2 (**C**) 0 (**D**) $\boxed{3}$ (**E**) 4

26. The equation $2^{x^2} = 16^{2x-3}$ has two solutions. What is their sum?

(**A**) -2 (**B**) 0 (**C**) 2 (**D**) 6 (**E**) $\boxed{8}$

27. A circle passes through the points $(0, 10)$, $(6, 4)$, and $(4, -2)$. What is the radius of the circle?

(**A**) 6 (**B**) $6\sqrt{2}$ (**C**) 7 (**D**) $7\sqrt{2}$ (**E**) $\boxed{\text{None of these}}$

28. How many integers does the solution set of $\dfrac{x^2-16}{x^2-4} \leq 0$ contain?

(**A**) $\boxed{4}$ (**B**) 6 (**C**) 7 (**D**) 9 (**E**) Infinitely many

29. If $\sin^2 x + \sin x \cos x - 6\cos^2 x = 0$ and $-\dfrac{\pi}{2} < x < 0$ what is the value of $\tan x$?

(**A**) 0 (**B**) $\boxed{-3}$ (**C**) 2 (**D**) -6 (**E**) None of these

30. How many distinguishable 7-letter words can you spell using all seven letters in the word $ALABAMA$? (The "words" do not have to make sense-for example, $AABAAML$ or $BAMAAAL$ are to be counted as seven letter words using all the letters in $ALABAMA$.)

(**A**) 24 (**B**) 105 (**C**) $\boxed{210}$ (**D**) 315 (**E**) 1260

31. What is the distance from the center of the ellipse $x^2 + 4y^2 + 2x - 8y + 1{=}0$ to the focus of the parabola $y^2 = 8x$?

(**A**) $\boxed{\sqrt{10}}$ (**B**) 6 (**C**) $\sqrt{5}$ (**D**) 4 (**E**) $\sqrt{2}$

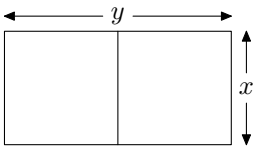
32. If $\sin\alpha = \dfrac{-5}{13}$ and α is in Quadrant III, what is the value of $\cos\left(\dfrac{\alpha}{2}\right)$?

(**A**) $-\dfrac{6}{13}$ (**B**) $\frac{2\sqrt{3}}{13}$ (**C**) $-\dfrac{\sqrt{5}}{12}$ (**D**) $\frac{4\sqrt{3}}{12}$ (**E**) $\boxed{-\dfrac{\sqrt{26}}{26}}$

33. Simplify $\dfrac{\sin x}{1-\cos x} - \dfrac{\sin x}{1+\cos x}$.

(**A**) $2\tan x$ (**B**) $\cos x - \sin x$ (**C**) $\boxed{2\cot x}$ (**D**) $\cos x + \sin x$ (**E**) None of these

34. A farmer wishes to use 600 feet to enclose a rectangular region and subdivide the region into two smaller rectangles, as shown in the figure. The total enclosed area is 15,000 square feet. How many feet should y be?



(**A**) 75 (**B**) 100 (**C**) 125 (**D**) $\boxed{150}$ (**E**) 175

35. Let $[x]$ denote the greatest integer less than or equal to x. Define the function f by $f(x) = [x] + [-x]$ If the number a is not an integer, and $n < a < n+1$ for some integer n, what is the value of $f(a)$?

(**A**) $2n-1$ (**B**) $\boxed{-1}$ (**C**) $2n$ (**D**) 0 (**E**) $2n+1$

36. The function f is an even function if $f(-x) = f(x)$ for all x in the domain of f. Which one of the following functions is even?

(**A**) $f(x) = (x-1)^2$ (**B**) $f(x) = \sin x$

(**C**) $f(x) = (x-1)^3$ (**D**) $\boxed{f(x) = \cos x}$ (**E**) None of these

37. How many solutions does the equation $\sqrt{x-3} - 1 = \sqrt{4-x}$ have?

(**A**) 0 (**B**) $\boxed{1}$ (**C**) 2 (**D**) 4 (**E**) Infinitely many

38. The composite of two functions f and g is denoted by $f \circ g$ and defined by $(f \circ g)(x) = f(g(x))$. When $f(x) = \dfrac{6x}{x-1}$ and $g(x) = \dfrac{5x}{x-2}$ which one of the following is equal to $(f \circ g)(x)$?

(**A**) $\dfrac{4-x}{x-2}$ (**B**) $\dfrac{30x}{5x+2}$ (**C**) $\dfrac{x-2}{4x+2}$ (**D**) $\boxed{\dfrac{15x}{2x+1}}$ (**E**) $\dfrac{x}{5x-2}$

39. Find the angular speed, in radians per second, of the second hand of a clock.

(**A**) $\dfrac{\pi}{5}$ (**B**) 0.2 (**C**) $\dfrac{\pi}{15}$ (**D**) 0.4 (**E**) $\boxed{\dfrac{\pi}{30}}$

40. Find the rectangular coordinates of the point whose polar coordinates are $(6, \dfrac{3\pi}{4})$.

(**A**) $\boxed{(-3\sqrt{2}, 3\sqrt{2})}$ (**B**) $(-2\sqrt{3}, 2\sqrt{3})$ (**C**) $(3\sqrt{2}, -3\sqrt{2})$ (**D**) $(2\sqrt{3}, -2\sqrt{3})$ (**E**) $(6, 135)$

41. Find $\tan\left(\arcsin\left(-\dfrac{4}{5}\right) - \arccos\left(-\dfrac{5}{13}\right)\right)$.

(**A**) $\dfrac{25}{63}$ (**B**) $-\dfrac{3}{7}$ (**C**) $\dfrac{4}{13}$ (**D**) $\dfrac{-33}{56}$ (**E**) $\boxed{\dfrac{16}{63}}$

42. Define $\eta(x, y, z) = \dfrac{x-z}{y-z}$. If $\eta(x, z, y) = -5$, what is the value of $\eta(x, y, z)$?

(**A**) -4 (**B**) -5 (**C**) $\boxed{6}$ (**D**) 4 (**E**) None of these

43. There are two numbers x making the value of the determinant shown equal to 86. What is the sum of these two numbers?

$$\begin{vmatrix} 1 & -2 & 5 \\ 2 & x & -1 \\ 0 & 4 & 2x \end{vmatrix}$$

(**A**) $\boxed{-4}$ (**B**) 5 (**C**) -3 (**D**) 9 (**E**) -1

44. Given that $\sin\alpha = -\dfrac{4}{5}$ and $\cos\beta = \dfrac{5}{13}$ with α in Quadrant III and β in Quadrant IV, find $\cos(\alpha + \beta)$.

(**A**) $\boxed{-\dfrac{63}{65}}$ (**B**) $\dfrac{5}{12}$ (**C**) $-\dfrac{33}{65}$ (**D**) $\dfrac{7}{12}$ (**E**) $-\dfrac{24}{65}$

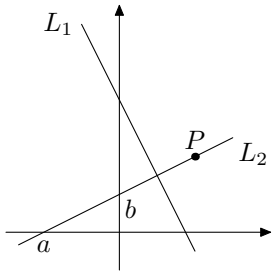
45. A family has six children (no twins). What is the probability that this family has three boys and three girls?

(**A**) $\dfrac{1}{2}$ (**B**) $\boxed{\dfrac{5}{16}}$ (**C**) $\dfrac{3}{8}$ (**D**) $\dfrac{7}{32}$ (**E**) $\dfrac{15}{64}$

46. What percent of the domain of the function $f(x) = \dfrac{\sqrt{9-x^2}}{\sqrt[4]{9-|2x+5|}}$ consists of positive numbers?

(**A**) 50% (**B**) 65% (**C**) $\boxed{40\%}$ (**D**) 75% (**E**) 30%

47. Line L_1 has equation $Ax + By = C$ where A, B, and C are positive real numbers. Line L_2 passes through the point $P = (1, 1)$ and is perpendicular to line L_1. Moreover, line L_2 crosses the coordinate axis at the numbers a and b as indicated in the figure. Find the number $a + b$.



(**A**) $1 - \frac{1}{A} - \frac{1}{B}$ (**B**) $\frac{A-B}{AB}$ (**C**) $1 - (\frac{A}{B})^2$

(**D**) $\frac{C}{A-B}$ (**E**) $\boxed{-\frac{(A-B)^2}{AB}}$

48. Let $f(x) = \frac{ax+b}{cx+d}$ where a, b, c, and d are positive real numbers and $ad - bc \neq 0$. Which one of the following numbers is not in the domain of f^{-1} (the inverse function of f)?

(**A**) $\boxed{\frac{a}{c}}$ (**B**) $\frac{1}{ad-bc}$ (**C**) $\frac{a-b}{c-d}$ (**D**) $\frac{1}{a-c}$ (**E**) $\frac{b-d}{ad-bc}$

49. 80% of all CopperTop batteries last for at least 10 hours of use. 15% last for at least 15 hours of use. What is the probability that a given CopperTop battery will last 15 hours, given that it has already lasted 10 hours?

(**A**) $\frac{4}{25}$ (**B**) $\frac{1}{8}$ (**C**) $\boxed{\frac{3}{16}}$ (**D**) $\frac{9}{40}$ (**E**) $\frac{1}{5}$

50. If $x + \frac{1}{x} = 4$ and $x < 2$ what is the number $x^2 + \sqrt{\frac{1}{x^2}}$?

(**A**) 16 (**B**) $8\sqrt{3}$ (**C**) $\boxed{14}$ (**D**) $16\sqrt{3}$ (**E**) 8

What You Can Do With A Mathematics Major

Occupational opportunities

Actuarial and Insurance	Government	Accountant
Computer & Information Sciences	Investment Analyst	Financial Planner
Researcher	Benefits Specialist	Mathematician
Demographers	Computer Programmer	Cartographer
Data Processor	Navigator	Meteorologist
Applications Programmer	Ecologist	Health
Systems Analyst	Biomedical Engineer	Bio-mathematician
Computer Applications Engineer	Operations Analyst	Operations Research
Control Systems Engineer	Control Systems Engineer	Systems Engineer
Numerical Analyst	Teaching	BusinessIndustry
Statistician	Engineering Analyst	Financial Analyst
Technical Writer	Homeland Security	Communications Engineer

Study in the field of mathematics offers an education with an emphasis on careful problem analysis, precision of thought and expression, and the mathematical skills needed for work in many other areas. Many important problems in government, private industry, health and environmental fields, and the academic world require sophisticated mathematical techniques for their solution. The study of mathematics provides specific analytical and quantitative tools, as well as general problem-solving skills, for dealing with these problems. The University of Alabama offers undergraduate and graduates degrees in Mathematics. Please visit **www.ua.edu** and refer to the undergraduate and graduate programs for additional information.

Engineering Math Advancement Program

The University of Alabama is offering a new summer program to build math skills for students entering engineering. The Engineering Math Advancement Program (EMAP) is a summer residence class that addresses math and engineering prerequisites for incoming engineering students. The program targets bright students who may not have retained the information learned in high school and provides and opportunity to hone technical abilities before entering college. The goal of E-MAP is to assisting entering freshmen in developing a solid background in calculus to succeed in engineering before they start at the University.

Classes are designed around Precalculus Algebra and Trigonometry and incorporate important learning principles to ensure that knowledge is retained and not just memorized. Students develop their skills through hands-on experiences, problem solving teaming exercises, and interaction with engineering professors and instructors through an interdisciplinary Living Laboratory program. Experiments allow students to use simple calculus in engineering applications. The program also involves introducing students to local practicing engineers through work on one or more community service engineering-related activities. E-MAP will reserve 33-40 percent of enrollment space for underrepresented groups. Financial assistance is available based on need. Please visit **emap.ua.edu** for additional information.