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08

ALABAMA

STATEWIDE MATHEMATICS CONTEST



First Round : March 29, 2008
 Second Round: April 26, 2008 at The University of Alabama

DIVISION II COMPREHENSIVE EXAM

Construction of this test directed
 by
 Robert Moore, The University of Alabama

INSTRUCTIONS

This test consists of 50 multiple choice questions. The questions have not been arranged in order of difficulty. For each question, choose the best of the five answer choices labeled A, B, C, D, and E.

The test will be scored as follows: 5 points for each correct answer, 1 point for each question left unanswered, and 0 points for each wrong answer. (Thus a “perfect paper” with all questions answered correctly earns a score of 250, a blank paper earns a score of 50, and a paper with all questions answered incorrectly earns a score of 0.)

Random guessing will not, on average, either increase or decrease your score. However, if you can eliminate one or more of the answer choices as wrong, then it is to your advantage to guess among the remaining choices.

- All variables and constants, except those indicated otherwise, represent real numbers.
- Diagrams are not necessarily to scale.

We use the following geometric notation:

- | | |
|--|--|
| • If A and B are points, then: | • If A is an angle, then: |
| \overline{AB} is the segment between A and B | $m \angle A$ is the measure of angle A in degrees |
| \overleftrightarrow{AB} is the line containing A and B | • If A and B are points on a circle, then: |
| \overrightarrow{AB} is the ray from A through B | \widehat{AB} is the arc between A and B |
| AB is the distance between A and B | $m \widehat{AB}$ is the measure of \widehat{AB} in degrees |

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What You Can Do With A Mathematics Major

Occupational opportunities

Actuarial and Insurance	Government	Accountant
Computer & Information Sciences	Investment Analyst	Financial Planner
Researcher	Benefits Specialist	Mathematician
Demographers	Computer Programmer	Cartographer
Data Processor	Navigator	Meteorologist
Applications Programmer	Ecologist	Health
Systems Analyst	Biomedical Engineer	Bio-mathematician
Computer Applications Engineer	Operations Analyst	Operations Research
Control Systems Engineer	Control Systems Engineer	Systems Engineer
Numerical Analyst	Teaching	Business Industry
Statistician	Engineering Analyst	Financial Analyst
Technical Writer	Homeland Security	Communications Engineer

Study in the field of mathematics offers an education with an emphasis on careful problem analysis, precision of thought and expression, and the mathematical skills needed for work in many other areas. Many important problems in government, private industry, health and environmental fields, and the academic world require sophisticated mathematical techniques for their solution. The study of mathematics provides specific analytical and quantitative tools, as well as general problem-solving skills, for dealing with these problems. The University of Alabama offers undergraduate and graduate degrees in Mathematics. Please visit www.ua.edu and refer to the undergraduate and graduate programs for additional information.

Engineering Math Advancement Program

The University of Alabama is offering a new summer program to build math skills for students entering engineering. The Engineering Math Advancement Program (EMAP) is a summer residence class that addresses math and engineering prerequisites for incoming engineering students. The program targets bright students who may not have retained the information learned in high school and provides an opportunity to hone technical abilities before entering college. The goal of E-MAP is to assist entering freshmen in developing a solid background in calculus to succeed in engineering before they start at the University.

Classes are designed around Precalculus Algebra and Trigonometry and incorporate important learning principles to ensure that knowledge is retained and not just memorized. Students develop their skills through hands-on experiences, problem solving teaming exercises, and interaction with engineering professors and instructors through an interdisciplinary Living Laboratory program. Experiments allow students to use simple calculus in engineering applications. The program also involves introducing students to local practicing engineers through work on one or more community service engineering-related activities. E-MAP will reserve 33-40 percent of enrollment space for underrepresented groups. Financial assistance is available based on need. Please visit emap.ua.edu for additional information.

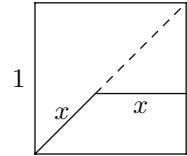
1. A cup contains 3 blue balls, 4 red balls and 1 white ball. 4 balls are chosen at random from the cup without replacement. What is the probability that *at least* one red ball is chosen?

(A) $\frac{59}{60}$ (B) $\frac{34}{35}$ (C) $\frac{1}{3}$ (D) $\frac{1}{2}$ (E) none of these

2. Solve for the positive root of the equation: $(\sqrt{200} + \sqrt{56})x^2 + 10x - 2(\sqrt{50} - \sqrt{14}) = 0$.

(A) $\frac{\sqrt{26}}{\sqrt{14}}$ (B) $\sqrt{200} - \sqrt{56}$ (C) $\frac{5\sqrt{2} - \sqrt{14}}{9}$ (D) $\frac{10}{\sqrt{200} - \sqrt{56}}$ (E) none of these

3. The large square in the picture is 1 inch on a side. The point on the diagonal is equally distant from the bottom left corner and the right side. What is that distance (in inches)?



(A) $2 - \sqrt{2}$ (B) $1 + \sqrt{2}$ (C) $\sqrt{2}$ (D) $\sqrt{2} - 1$ (E) None of these

4. Evaluate: $\cos 67^\circ \cos 22^\circ + \cos 23^\circ \cos 68^\circ$.

(A) $\sqrt{2}$ (B) $\cos 89^\circ$ (C) 0 (D) $1/\sqrt{2}$ (E) none of these

5. On planet Tris, which circles the star Poly, the natives use length units of dee's and doe's; one dee equals seven does. They don't use square units to measure area, but triangular units: this works fine, since triangles can tile the plane, and one triangular dee is exactly an equilateral triangle whose side is one dee. How many triangular does are there in a triangular dee?

(A) 9 (B) 3 (C) $3\sqrt{3}$ (D) 7 (E) none of these

6. How many ways are there to rearrange the letters of the word SUSURRUS?

(A) 8! (B) $\frac{8!}{5!3!}$ (C) $\binom{8}{3}\binom{5}{3}$ (D) $\binom{8}{5}$ (E) none of these

7. A large square is drawn, with area A . A circle is inscribed in the square and a smaller square is inscribed in the circle. What is the area of the smaller square?

(A) $A/2$ (B) $A/4$ (C) $(1/\sqrt{2})A$ (D) \sqrt{A} (E) none of these

8. Town B is 3 miles west of town A . Town C is 2 miles north of town B . Town D is 3 miles northeast of town C . How far (in miles) is town D from town A ?

(A) $\sqrt{\frac{35}{2} - 3\sqrt{2}}$ (B) $22 - 3\sqrt{2}$ (C) $\sqrt{22 - 3\sqrt{2}}$ (D) 8 (E) none of these

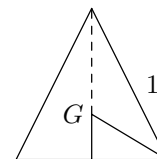
9. Evaluate the following, given that an answer exists and is less than e :

$$\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\dots}}}$$

(A) 4 (B) $\sqrt{2}$ (C) 2 (D) $2\sqrt{2}$ (E) none of these

10. Point G is the centroid of the equilateral triangle. What is the area of the small triangle at the lower right in the picture?

(A) $\sqrt{3}$ (B) $\sqrt{2}$ (C) $\frac{1}{2}\sqrt{3}$ (D) $\frac{1}{3}\sqrt{3}$ (E) none of these



11. Simplify: $\sqrt{28 + \sqrt{300}}$.

(A) 5 (B) $25 + \sqrt{3}$ (C) $5 + \sqrt{3}$ (D) $5 + 2\sqrt{3}$ (E) none of these

12. In a large lecture class, 63 of the students are freshmen and 46 are math majors. 8 students are neither math majors nor freshmen. How many students are in the class?

(A) 109 (B) 117 (C) 101
 (D) There is not enough information to solve the problem (E) none of these

13. In Professor Wu's latest research paper he defines the concepts of *quenk* and *vetzal* and he proves the Fundamental Theorem of Quenks:

Theorem: There is no quenk that is not a vetzal.

A statement that is *completely equivalent* to this theorem is:

(A) Every vetzal is a quenk (B) Every quenk is a vetzal (C) Some vetzals are not quenks
 (D) Some quenks are not vetzals (E) None of these statements is equivalent to Wu's theorem

14. On Tuesday, a merchant remarks all his prices, increasing them by 25%. On Wednesday he puts a sign in the window saying "SALE!!! Every item 25% off marked price!!!" What is the result of these changes?

I. Wednesday's prices are the same as Monday's.
 II. Wednesday's prices are 25% less than Monday's.
 III. Wednesday's prices are 93.75% of Monday's.
 IV. Wednesday's prices are about 106.25% of Monday's

(A) I (B) II (C) III (D) IV (E) none of these

15. a is a real number larger than 1. Simplify the expression: $(2^{-\log_a(a^3)}) \cdot (a^{-\log_a(1/2)})$.

(A) $\log_a 2$ (B) $\frac{1}{4}$ (C) $\frac{1}{2}$ (D) $2a$ (E) none of these

16. Solve the equation:

$$\begin{vmatrix} 1 & 1 \\ 2x & 3x \end{vmatrix} = x$$

where the vertical bars mean the determinant.

(A) $x = 0$ (B) $x = 1$ (C) $x = 0$ or $x = 1$
 (D) Every value of x satisfies this equation (E) none of these

17. If x is very close to 1 — but not equal to 1 — then the expression

$$\frac{x^2 - 1}{x - 1}$$

is very close to what?

- (A) 0 (B) 1 (C) (D) undefined (E) none of these
18. A plumber wants to replace an old 6-inch diameter pipe by some newer pipes, which are only 1 inch in diameter. How many of the new pipes will he need to get the same cross-sectional area?
- (A) 6 (B) 6π (C) 36π (D) (E) none of these
19. A triangle has sides of length 24, 10, and 26. What is the radius of the inscribed circle?
- (A) 26 (B) (C) 3 (D) 8 (E) none of these
20. The price of blue socks is twice the price of black socks. If John buys 20 pairs of black socks and 30 pairs of blue socks he pays \$120. How much does one pair of blue socks cost?
- (A) \$1.50 (B) (C) \$4.50 (D) \$1.00 (E) none of these
21. If $\tan \phi = \frac{4}{3}$, then what is $\tan 3\phi$?

- (A) (B) $\frac{171}{172}$ (C) $\frac{24}{25}$ (D) $\frac{25}{24}$ (E) none of these

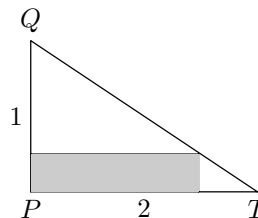
22. You are given the equation

$$C = \frac{en}{R + nr}$$

where e , R and r represent positive constants. Suppose that n starts with value 0 and increases. Then the value of C :

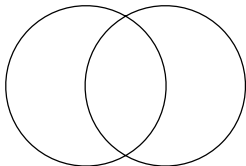
- (A) (B) decreases always (C) first increases, then decreases
 (D) first decreases, then increases (E) remains constant
23. The rectangle shown is inscribed in the right triangle with sides lengths 1 and 2. What is the largest possible area of the rectangle?

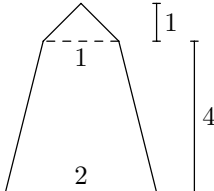
- (A) 1 (B) (C) 1/3 (D) 1/4 (E) none of these



24. The equation $x + \sqrt{x - 2} = 4$ has:

- (A) two real roots (B) one real and one purely imaginary root (C)
 (D) two complex roots, neither of which is purely imaginary (E) none of these

25. The front face of a rectangular box has area A_f . The area of the right side of the box is A_s and the area of the top is A_t . The product $A_f A_s A_t$ represents:
- (A) the square of the surface area of the box (B) the total surface area of the box
 (C) the square of the volume of the box (D) the volume of the box (E) none of these
26. Line A is measured to be 10 inches long, with a possible relative error of $\pm 0.2\%$. Line B is measured to be 100 inches long, with a possible absolute error of ± 0.2 inches. Compared to the absolute error in the measurement of line A , the absolute error in line B is
- (A) greater by 0.18 inches (B) greater by 1.8 inches
 (C) a hundred times greater (D) exactly the same (E) none of these
27. A regular hexagon is inscribed in a circle of radius r . What is the *perimeter* of the hexagon?
- (A) $6r$ (B) $3r$ (C) $6\pi r$ (D) $3\pi r$ (E) none of these
28. Two circles of radius 1 are drawn so that each circle passes through the center of the other. What is the *perimeter* of the region of overlap?
- (A) $\pi/3$ (B) $2\pi/3$ (C) $4\pi/3$ (D) $4/3$ (E) none of these
- 
29. What is the last digit of 2^{2008} ?
- (A) 2 (B) 4 (C) 6 (D) 8 (E) none of these
30. A convex polygon has 100 sides. How many different *diagonals* can be drawn in this figure?
- (A) 4850 (B) 4950 (C) 9900 (D) 8800 (E) none of these
31. Start with an equilateral triangle of side 3. Form a new triangle by joining the midpoints of the sides. Then form a third triangle by joining the midpoints of the sides of the second triangle. Continue in this fashion. The sum of the perimeters of all the triangles is:
- (A) infinite (B) 6 (C) 13.5 (D) 15.75 (E) none of these
32. At 2 : 15 pm, the hands of a clock form an angle of:
- (A) 30° (B) 5° (C) 22.5° (D) 7.5° (E) none of these
33. Two cards are available. One card is red on both sides and one card is red on one side and blue on the other. One of the cards is chosen at random and placed on the table. The visible side is red. What is the probability that the other side is red?
- (A) $1/2$ (B) $1/3$ (C) $2/3$ (D) $3/4$ (E) none of these

34. If $\frac{2}{3}$ of a can of paint will cover $\frac{3}{5}$ of a wall, how much of a can of paint is required to cover $\frac{3}{4}$ of the wall?
- (A) $\frac{5}{6}$ (B) $\frac{6}{5}$ (C) $\frac{9}{30}$ (D) $\frac{10}{9}$ (E) none of these
35. A circular plot of grass 12 feet in diameter is divided by a straight gravel path 3 feet wide. One edge of the gravel path passes through the center of the grass plot. What is the area of the grass in the plot (in square feet)?
- (A) $36\pi - 34$ (B) $35\pi - 9\sqrt{3}$ (C) $30\pi - 15$ (D) $36\pi - 33$ (E) none of these
36. If $i^2 = -1$, what is the value of $(1 + i)^{16} - (1 - i)^{16}$?
- (A) 0 (B) 1 (C) 2^8 (D) -2^8 (E) none of these
37. Which is the largest of the following numbers?
- (A) $3^{6 \cdot 9}$ (B) $6^{3 \cdot 9}$ (C) $9^{6 \cdot 3}$ (D) $(3 \cdot 6)^9$ (E) none of these
38. Three girls and three boys are seated around a circular table with six seats. They must alternate: boy, girl, boy, girl, etc. Ignoring rotations, how many different ways are there to seat the people? (Two configurations are different if there are two people who are neighbors in one configuration but not the other.)
- (A) 3 (B) 6 (C) 36 (D) 12 (E) none of these
39. Which of the following is equal to $\sqrt{4 + 2\sqrt{3}}$?
- (A) $\sqrt{11}$ (B) $1 + \sqrt{3}$ (C) $2 + \sqrt{3}$ (D) $-1 - \sqrt{3}$ (E) none of these
40. 16 college students are going to the beach in 4 identical vans. Each van can hold exactly four students. How many ways can we distribute the students in the vans? (Two distributions are different if there are two students who ride together in one distribution but not in the other. If the same groups of students are together, it does not matter which van they ride in.)
- (A) $16!$ (B) $(4!)^3$ (C) $\binom{16}{4}^3$ (D) $\binom{16}{4}\binom{12}{4}\binom{8}{4}$ (E) none of these
41. Speaking of obelisks, what is the area of the figure below in square units?
- (A) 5.5 (B) 6.0 (C) 6.5 (D) 7.0 (E) none of these
- 
42. In a race over d yards, A would beat B by 20 yards; B would beat C by 10 yards; and A would beat C by 28 yards. What is the value of d ?
- (A) 100 yd (B) 58 yd (C) 116 yd
(D) not enough information to answer the question (E) none of these

43. For points (x, y) on the line $2x + 3y = 6$, what is the *smallest value* of the quantity $\sqrt{x^2 + y^2}$?
- (A) $\frac{6\sqrt{13}}{13}$ (B) 6 (C) $\frac{1}{2}\sqrt{13}$ (D) $\sqrt{13}$ (E) none of these
44. Five real numbers are chosen and put in order from smallest to largest. The average of all five is 14. The average of the three middle numbers is only 13. What is the average of the largest and smallest?
- (A) 15 (B) $15\frac{1}{2}$ (C) 16 (D) $16\frac{1}{2}$ (E) none of these
45. a is a positive real number and e is the base for the natural logarithm. Simplify the expression: $\log_a(b^{e \cdot \ln a})$.
- (A) $\frac{e}{\ln b}$ (B) $e \cdot \frac{\ln b}{\ln a}$ (C) $e \cdot \frac{\ln a}{\ln b}$ (D) $\left(\frac{\ln b}{\ln a}\right)^e$ (E) none of these
46. A right circular cone and a right circular cylinder have the same height and the same base. Compared to the cone, how large is the volume of the cylinder?
- (A) 4 times as large as the volume of the cone (B) 2 times as large as the volume of the cone
(C) $3/2$ times the volume of the cone (D) $\sqrt{3}$ times the volume of the cone (E) none of these
47. We just blew some air into a spherical balloon and doubled its volume. By how much did we multiply the *surface area*?
- (A) 2 (B) $2^{1/3}$ (C) $2^{2/3}$ (D) 2^3 (E) none of these
48. Let f be a function that maps the interval $(0, \infty)$ into itself. We will say that f is “damping” if, for all $x > y > 0$, we have
- $$\frac{x}{y} > \frac{f(x)}{f(y)}$$
- Which of the following must be true, for any such function?
- (A) $\ln f(x)$ is increasing (B) $\ln f(x)$ is decreasing (C) $\ln x - \ln f(x)$ is increasing
(D) $\ln x - \ln f(x)$ is decreasing (E) There are no such functions
49. I have Q quarters and D dimes in my pocket, whose total value is \$2.10. Which of the following equations must be true?
- (A) $Q + D = 210$ (B) $Q + D = 21$ (C) $10Q + 25D = 210$
(D) $25Q + 10D = 21$ (E) none of these
50. My mother’s iced tea recipe uses a certain amount of sugar to make 1 gallon of tea. But I like it less sweet so I only use $4/5$ of the sugar it calls for. Last night I made 3 and $3/4$ gallons of tea. My sister is a sugar freak and uses 1 and $1/2$ times as much sugar as the recipe calls for. Last night she made 2 and $1/4$ gallons of tea. Who used more sugar?
- (A) I used more sugar (B) My sister used more sugar (C) We used the same amount of sugar
(D) The answer depends on the call of the original recipe (E) none of these